MATHEMATICS 271 L01 FALL 2004
ASSIGNMENT 5 SOLUTION

1. Let A and B be nonempty sets. Let $f : A \to B$ be a function. Let $\mathcal{S}$ be a relation on B. Let $\mathcal{R}$ be the relation on A defined by:

$(a, b) \in \mathcal{R}$ *if and only if* $(f(a), f(b)) \in \mathcal{S}$.

Let $\mathcal{A}$ be the statement: "if $\mathcal{S}$ is an equivalence relation on B then $\mathcal{R}$ is an equivalence relation on A."

(a) Prove that $\mathcal{A}$ is true.

(b) Prove that the converse of $\mathcal{A}$ is false.

(c) Prove that if f is onto B then the converse of $\mathcal{A}$ is true.

(d) Now, suppose that $A = B = \mathbb{Z}^+$, the set of all positive integers, and $\mathcal{S}$ is the relation "congruence modulo 5". Then, by part (a), $\mathcal{R}$ is an equivalence relation on $A = \mathbb{Z}^+$. Find a one-to-one function $f : A \to B$ so that $\mathcal{R}$ has exactly two equivalence classes

Solution:

(a) Suppose that $\mathcal{S}$ is an equivalence relation on B. We prove that $\mathcal{R}$ is an equivalence relation on A.

First, we prove that $\mathcal{R}$ is reflexive. Let $x \in A$. Since $f(x) \in B$ and $\mathcal{S}$ is reflexive on B, we have $(f(x), f(x)) \in \mathcal{S}$, and so $(x, x) \in \mathcal{R}$ by the definition of $\mathcal{R}$.

Next, we prove that $\mathcal{R}$ is symmetric. Let $x, y \in A$ and suppose that $(x, y) \in \mathcal{R}$. Since $(x, y) \in \mathcal{R}$, we have $(f(x), f(y)) \in \mathcal{S}$. Since $(f(x), f(y)) \in \mathcal{S}$ and $\mathcal{S}$ is symmetric, we have $(f(y), f(x)) \in \mathcal{S}$, and so $(y, x) \in \mathcal{R}$ by the definition of $\mathcal{R}$.

Lastly, we prove that $\mathcal{R}$ is transitive. Let $x, y, z \in A$ and suppose that $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$. Since $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, we have $(f(x), f(y)) \in \mathcal{S}$ and $(f(y), f(z)) \in \mathcal{S}$. Since $(f(x), f(y)) \in \mathcal{S}$ and $(f(y), f(z)) \in \mathcal{S}$,.and $\mathcal{S}$ is transitive, we have $(f(x), f(z)) \in \mathcal{S}$, and so $(x, z) \in \mathcal{R}$ by the definition of $\mathcal{R}$.

Since $\mathcal{R}$ is reflexive, symmetric and transitive, $\mathcal{R}$ is an equivalence relation on A.

(b) The converse of $\mathcal{A}$ is false because when $A = \{1\}$, $B = \{1, 2\}$, $f : A \to B$ defined by $f(1) = 1$ and $\mathcal{S} = \{(1, 1)\}$, we have $\mathcal{R} = \{(1, 1)\}$. and therefore $\mathcal{R}$ is an equivalence relation on A, but $\mathcal{S}$ is not an equivalence relation on B.

(c) Suppose that $f : A \to B$ is an onto function, and suppose that $\mathcal{R}$ is an equivalence relation on A. We show that $\mathcal{S}$ is an equivalence relation on B.

First, we prove that $\mathcal{S}$ is reflexive. Let $x \in B$. Since $x \in B$ and f is onto, there is $u \in A$ so that $f(u) = x$. Since $u \in A$ and $\mathcal{R}$ is reflexive on A, we have $(u, u) \in \mathcal{R}$, and so $(x, x) = (f(u), f(u)) \in \mathcal{S}$ by the definition of $\mathcal{R}$.

Next, we prove that $\mathcal{S}$ is symmetric. Let $x, y \in B$ and suppose that $(x, y) \in \mathcal{S}$. Since $x, y \in B$ and f is onto, there are $s, t \in A$ so that $f(s) = x$ and $f(t) = y$. Since $(f(s), f(t)) = (x, y) \in \mathcal{S}$, we have $(s, t) \in \mathcal{R}$. Since $(s, t) \in \mathcal{R}$ and $\mathcal{R}$ is symmetric, we have $(t, s) \in \mathcal{R}$, and so $(f(t), f(s)) \in \mathcal{S}$ by the definition of $\mathcal{R}$, that is, $(y, x) \in \mathcal{S}$.

Lastly, we prove that $\mathcal{S}$ is transitive. Let $x, y, z \in B$ and suppose that $(x, y) \in \mathcal{S}$ and $(y, z) \in \mathcal{S}$. Since $x, y, z \in B$ and f is onto, there are $r, s, t \in B$ so that $f(r) = x$, $f(s) = y$ and $f(t) = z$. Since $(x, y) = (f(r), f(s)) \in \mathcal{S}$ and $(y, z) = (f(s), f(t)) \in \mathcal{S}$, we have

$(r,s) \in \mathcal{R}$ and $(s,t) \in \mathcal{R}$. Since $(r,s) \in \mathcal{R}$ and $(s,t) \in \mathcal{R}$,.and $\mathcal{R}$ is transitive, we have $(r,t) \in \mathcal{R}$, and so $(x,z) = (f(r), f(t)) \in \mathcal{S}$ by the definition of $\mathcal{R}$.

Since $\mathcal{S}$ is reflexive, symmetric and transitive, $\mathcal{R}$ is an equivalence relation on A.

(d) Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the function defined by:

$$f(x) = \begin{cases} 5x & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

First, we prove that f is one-to-one. Let $x, y \in \mathbb{Z}^+$ and suppose that $f(x) = f(y)$. We have two cases:

Case 1: $x = 1$. Then $f(x) = f(y) = f(1) = 1$. Then y must be 1. For otherwise, from $f(y) = 1$ we have $5y = 1$ and so $y = \frac{1}{5}$, which is not an integer. Thus, $y = 1 = x$.

Case 2: $x \neq 1$. Then from $f(x) = f(y)$ we get $f(y) = 5x$ and so $y \neq 1$. For otherwise, from $y = 1$ and $f(y) = 5x$, we get $1 = 5x$ which is not possible since $x \in \mathbb{Z}^+$. Thus, $y \neq 1 \neq x$ and from $f(x) = f(y)$, we get $5x = 5y$ which implies $x = y$.

Next, we show that $\mathcal{R}$ has exactly two equivalence classes which are $[2] = \{x \in \mathbb{Z}^+ \mid x \neq 1\}$ and $[1] = \{1\}$.

First, we show $[2] = \{x \in \mathbb{Z}^+ \mid x \neq 1\}$. Let $x \in [2] = \{x \in \mathbb{Z}^+ \mid x\mathcal{R}2\}$, that is, $f(x) \equiv 10 \equiv 0 \pmod{n}$ which means $f(x)$ is divisible by 5, and so $x \neq 1$. Thus, $[2] \subseteq \{x \in \mathbb{Z}^+ \mid x \neq 1\}$. Next, let $x \in \mathbb{Z}^+$ so that $x \neq 1$. Then $f(x) = 5x \equiv 0 \equiv 10 \pmod{n}$, that is, $f(x) \equiv f(2) \pmod{n}$ and so $\{x \in \mathbb{Z}^+ \mid x \neq 1\} \subseteq [2]$. Thus, $[2] = \{x \in \mathbb{Z}^+ \mid x \neq 1\}$.

Similarly, it is easy toshow that $[1] = \{1\}$.

2. Let a and n be integers with $n \geq 2$. Define the relation $\mathcal{R}_{a,n}$ on $\mathbb{Z}$ by $(x,y) \in \mathcal{R}_{a,n}$ if and only if $ax \equiv ay \pmod{n}$.

(a) Prove that $\mathcal{R}_{a,n}$ is an equivalence relation on $\mathbb{Z}$.

(b) Show that for all integers $n \geq 2$, there is an integer $a \neq 0$ so that $\mathcal{R}_{a,n}$ has only one equivalence class.

(c) Show that there are integers a and n, where $n \geq 2$, so that $\mathcal{R}_{a,n}$ has exactly two equivalence classes.

(d) Suppose that a and n are relatively prime. Prove that $(x,y) \in \mathcal{R}_{a,n}$ if and only if $x \equiv y \pmod{n}$.

(e) Prove that if a and n are relatively prime then $\mathcal{R}_{a,n}$ has n equivalence classes.

Solution:

(a) From 1(a) with $\mathcal{R} = \mathcal{R}_{a,n}$, $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = ax$ for all $x \in \mathbb{Z}$, and $\mathcal{S}$ is the relation "congruence modulo n", we see that $\mathcal{R}_{a,n}$ is an equivalence relation on $\mathbb{Z}$.

(b) Let $a = n$, then for any $x, y \in \mathbb{Z}$, we have $f(x) - f(y) = nx - ny = n(x - y)$ which means $n \mid [f(x) - f(y)]$ and so $f(x) \equiv f(y) \pmod{n}$. This implies $x\mathcal{R}y$. Thus, $x\mathcal{R}y$ for all $x, y \in \mathbb{Z}$ and therefore, $\mathcal{R}_{a,n}$ has only one equivalence class.

(c) Let $a = 2$ and $n = 4$. Then $\mathcal{R}_{a,n}$ has exactly two equivalence classes, namely,.$[0] = \{x \in \mathbb{Z} \mid x \text{ is even}\}$ and $[1] = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$.

(d) Suppose that a and n are relatively prime. We prove that $(x,y) \in \mathcal{R}_{a,n}$ if and only if $x \equiv y \pmod{n}$.

($\Rightarrow$) Let $(x,y)\in\mathcal{R}_{a,n}$. We show that $x\equiv y\,(\mathrm{mod}\,n)$. Since $(x,y)\in\mathcal{R}_{a,n}$, we have $ax\equiv ay\,(\mathrm{mod}\,n)$. Since a and n are relatively prime, there is an integer b so that $ab\equiv 1\,(\mathrm{mod}\,n)$. Now, since $1\equiv ab\,(\mathrm{mod}\,n)$ and $x\equiv x\,(\mathrm{mod}\,n)$, we have $x\equiv abx\,(\mathrm{mod}\,n)$ by Theorem 10.4.3. From the same theorem, from $ax\equiv ay\,(\mathrm{mod}\,n)$ and $b\equiv b\,(\mathrm{mod}\,n)$, we have $abx\equiv aby\,(\mathrm{mod}\,n)$. Similarly, we can show that $aby\equiv y\,(\mathrm{mod}\,n)$. Next, from $x\equiv abx\,(\mathrm{mod}\,n)$, $abx\equiv aby\,(\mathrm{mod}\,n)$ and $aby\equiv y\,(\mathrm{mod}\,n)$, we get $x\equiv y\,(\mathrm{mod}\,n)$ by transitivity of congruence modulo n.

($\Longleftarrow$) Suppose that $x\equiv y\,(\mathrm{mod}\,n)$. Since $x\equiv y\,(\mathrm{mod}\,n)$ and $a\equiv a\,(\mathrm{mod}\,n)$, we get $ax\equiv ay\,(\mathrm{mod}\,n)$ and so $(x,y)\in\mathcal{R}_{a,n}$.

(e) From part (d), we see that $\mathcal{R}_{a,n}$ is the relation"congruence modulo n" and so it has n equivalence classes.

3. Let $A=\{1,2,3,4,5,6,7,8\}$. Give reasons to support your answers.

(a) Is it true that there is a relation $\mathcal{R}$ on A so that $\mathcal{R}$ is not symmetric, not antisymmetric but $\mathcal{R}$ is transitive?

(b) Is it true that there is a relation $\mathcal{S}$ on A so that $\mathcal{S}$ is symmetric, antisymmetric but $\mathcal{S}$ is not transitive?

(c) How many symmetric relations on A are there?

(d) How many antisymmetric relations on A are there?

(e) How many relations on A are there that are antisymetric or symmetric?

Solution:

(a) Yes, there is a relation $\mathcal{R}$ on A so that $\mathcal{R}$ is not symmetric, not antisymmetric but $\mathcal{R}$ is transitive. For example, let $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$. Then $\mathcal{R}$ is a relation on A which is not symmetric, not antisymmetric but it is transitive. $\mathcal{R}$ isnot symmetric because $(1,2)\in\mathcal{R}$ but $(2,1)\notin\mathcal{R}$. Next, $\mathcal{R}$ is not antisymmetric because $(2,3)\in\mathcal{R}$ and $(3,2)\in\mathcal{R}$, but $2\neq 3$. Lastly, to prove that $\mathcal{R}$ is transitive we suppose $x,y,z\in A$ so that $(x,y)\in\mathcal{R}$ and $(y,z)\in\mathcal{R}$, and we show that $(x,z)\in\mathcal{R}$. Since $(x,y)\in\mathcal{R}$ and $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$, we have six cases.

Case 1: $(x,y)=(1,2)$. Then $x=1$ and $y=2$. Then from $(y,z)\in\mathcal{R}$,we have $(2,z)\in\mathcal{R}$, so $z=3$ or 2 because $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$. In either case, we have $(x,z)\in\mathcal{R}$ because $(1,3)\in\mathcal{R}$ and $(1,2)\in\mathcal{R}$.

Case 2: $(x,y)=(2,3)$. Then $x=2$ and $y=3$. Then from $(y,z)\in\mathcal{R}$,we have $(3,z)\in\mathcal{R}$, so $z=3$ or 2 because $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$. In either case, we have $(x,z)\in\mathcal{R}$ because $(2,3)\in\mathcal{R}$ and $(2,2)\in\mathcal{R}$.

Case 3: $(x,y)=(3,2)$. Then $x=3$ and $y=2$. Then from $(y,z)\in\mathcal{R}$,we have $(2,z)\in\mathcal{R}$, so $z=3$ or 2 because $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$. In either case, we have $(x,z)\in\mathcal{R}$ because $(3,3)\in\mathcal{R}$ and $(3,2)\in\mathcal{R}$.

Case 4: $(x,y)=(1,3)$. Then $x=1$ and $y=3$. Then from $(y,z)\in\mathcal{R}$,we have $(3,z)\in\mathcal{R}$, so $z=3$ or 2 because $\mathcal{R}=\{(1,2),(2,3),(3,2),(1,3),(2,2),(3,3)\}$. In either case, we have $(x,z)\in\mathcal{R}$ because $(1,3)\in\mathcal{R}$ and $(1,2)\in\mathcal{R}$.

Case 5: $(x,y)=(2,2)$. Then $x=y=2$. Then since $x=y$ and $(y,z)\in\mathcal{R}$,we have $(x,z)\in\mathcal{R}$.

Case 6: $(x, y) = (3, 3)$. Then $x = y = 3$. Then since $x = y$ and $(y, z) \in \mathcal{R}$,we have $(x, z) \in \mathcal{R}$.

Thus, $\mathcal{R}$ is transitive.

(b) No, there is no relations $\mathcal{S}$ on A so that $\mathcal{S}$ is symmetric, antisymmetric but $\mathcal{S}$ is not transitive. We prove this by a contradiction proof. Suppose that there is a relation $\mathcal{S}$ on A so that $\mathcal{S}$ is symmetric, antisymmetric but $\mathcal{S}$ is not transitive. Since $\mathcal{S}$ is not transitive, there are $x, y, z \in A$ so that $(x, y) \in \mathcal{S}$, $(y, z) \in \mathcal{S}$ but $(x, z) \notin \mathcal{S}$. Since $(x, y) \in \mathcal{S}$ and $\mathcal{S}$ is symmetric, we get $(y, x) \in \mathcal{S}$. Now, since $(x, y) \in \mathcal{S}$ and $(y, x) \in \mathcal{S}$, we get $x = y$ by antisymmetry of $\mathcal{S}$. Since $x = y$ and $(y, z) \in \mathcal{S}$, we get $(x, z) \in \mathcal{S}$ which contradicts $(x, z) \notin \mathcal{S}$.

Thus, there is no relations $\mathcal{S}$ on A so that $\mathcal{S}$ is symmetric, antisymmetric but $\mathcal{S}$ is not transitive.

(c) The answer to this question is $2^8 \times 2^{\binom{8}{2}} = 2^8 \times 2^{28} = 2^{36}$.We note that A has $\binom{8}{2} = 28$ two element subsets. To construct a symmetric relation $\mathcal{R}$ on A, we have 2 choices for each element $a \in A$ (either we decide that $(a, a) \in \mathcal{R}$ or $(a, a) \notin \mathcal{R}$), and for each two element subset $\{a, b\}$ we have two choices (either we decide that $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, or we decide that $(a, b) \notin \mathcal{R}$ and $(b, a) \in \mathcal{R}$). Since A has 8 elements and $\binom{8}{2}$ two element subsets, we see that there are $2^8 \times 2^{\binom{8}{2}}$ symmetric relations on A.

(d) The answer to this question is $2^8 \times 3^{\binom{8}{2}} = 2^8 \times 3^{28}$. To construct an antisymmetric relation $\mathcal{R}$ on A, we have 2 choices for each element $a \in A$ (either we decide that $(a, a) \in \mathcal{R}$ or $(a, a) \notin \mathcal{R}$), and for each two element subset $\{a, b\}$ we have three choices (either we decide that $(a, b) \in \mathcal{R}$, or we decide that $(b, a) \in \mathcal{R}$, or we decide that $(a, b) \notin \mathcal{R}$ and $(b, a) \notin \mathcal{R}$). Since A has 8 elements and $\binom{8}{2}$ two element subsets, we see that there are $2^8 \times 3^{\binom{8}{2}}$ antisymmetric relations on A.

(e) The answer to this question is the number of symmetric relations on A plus the number of antisymmetric relations on A minus the number of relations on A which are both symmetric and antisymmetric which is $2^{36} + 2^8 \times 3^{28} - 2^8$. Thus, we only have to show that the number of relations on A which are both symmetric and antisymmetric is 2^8. Let $\mathcal{R}$ be a relation on A which is both symmetric and antisymmetric. Then for any $a \neq b \in A$, we can not have $(a, b) \in \mathcal{R}$. This is because if $(a, b) \in \mathcal{R}$.then $(b, a) \in \mathcal{R}$.by symmetry of $\mathcal{R}$ but then $a = b$ by antisymmetry of $\mathcal{R}$. From this observation, we see that the relations on A that are both symmetric and anti symmetric must be subsets of $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)\}$ and so there are 2^8 such relations.