MATHEMATICS 271 L01 FALL 2004
QUIZ 3 SOLUTION

1. Prove that
$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for all integers $n \geq 0$.
Solution:
Basis: $(n = 0)$
$\sum_{i=1}^{0+1} i \cdot 2^i = \sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2 = 0 + 2 = 0.2^{0+2} + 2$
Inductive step: Let $k \geq 0$ be an integer and suppose that
$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2.$ [IH]

We show that $\sum_{i=1}^{k+2} i \cdot 2^i = k \cdot 2^{k+3} + 2.$

Now,

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= \left(\sum_{i=1}^{k+1} i \cdot 2^i\right) + (k+2) \cdot 2^{k+2} \\
&= k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} \qquad \text{by [IH]} \\
&= (2k+2) \cdot 2^{k+2} + 2 \\
&= (k+) \cdot 2 \cdot 2^{k+2} + 2 \\
&= k \cdot 2^{k+3} + 2.
\end{aligned}$$

Thus, we have proved by induction that $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for all integers $n \geq 0$.

2. Prove that
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for all integers $n \geq 2$.
Solution:
Basis: $(n = 2)$
Since $\sqrt{2} > 1$, we have $\sqrt{2} + 1 > 2$ and so
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{2}{\sqrt{2}} = \sqrt{2}$
Inductive step: Let $k \geq 2$ be an integer and suppose that
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}.$ [IH]
We show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$
Now,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}} &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&> \sqrt{k} + \frac{1}{\sqrt{k+1}} \qquad \text{by [IH]} \\
&= \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&> \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} \\
&= \frac{k+1}{\sqrt{k+1}} \\
&= \sqrt{k+1}.
\end{aligned}$$

Thus, we have proved by induction that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for all integers $n \geq 2$.