MATHEMATICS 271 L01 FALL 2004
QUIZ 5 SOLUTON

1. Define the function $h : \mathbb{Z} \to \mathbb{Z}$ by putting $h(x) = 3x^2 - x$ for each $x \in \mathbb{Z}$.
(a) Is h one-to-one? Prove your answer.
Solution: h is one-to-one and here is a proof. Let $x, y \in \mathbb{Z}$ and suppose that $h(x) = h(y)$. Now,

$$\begin{aligned} h(x) = h(y) &\Leftrightarrow 3x^2 - x = 3y^2 - y \\ &\Leftrightarrow 3x^2 - x - 3y^2 + y = 0 \\ &\Leftrightarrow 3(x^2 - y^2) - (x - y) = 0 \\ &\Leftrightarrow 3(x-y)(x+y) - (x-y) = 0 \\ &\Leftrightarrow (x-y)[3(x+y) - 1] = 0 \qquad (\bigstar) \end{aligned}$$

Next, we show that $3(x+y) - 1 \neq 0$ by a contradiction proof. Suppose that $3(x+y) - 1 = 0$. It follows that $x + y = \frac{1}{3}$, which is impossible because x and y are integers.

Thus, $3(x+y) - 1 \neq 0$ and so from $(\bigstar)$ we get that $x - y = 0$ and so $x = y$. Hence, h is one-to-one.

(b) Is h onto? Prove your answer.
Solution: h is not onto. We prove that for all $x \in \mathbb{Z}$, $h(x) \neq 1$. Let $x \in \mathbb{Z}$, we prove that $h(x) \neq 1$ by a contradiction proof. Suppose that $h(x) = 1$, that is, $3x^2 - x = 1$. Thus, $x(3x - 1) = 1$, so x is a divisor of 1, and therefore, $x = 1$ or $x = -1$. However, when $x = 1$, $h(x) = h(1) = 2$ which contradicts the assumption that $h(x) = 1$, and when $x = -1$, $h(x) = h(1) = 4$ which contradicts the assumption that $h(x) = 1$. Thus, $h(x) \neq 1$ for all $x \in \mathbb{Z}$, so h is not onto.

[8] **2**. Let $A = \{1, 2, 3, 4\}$. Prove or disprove the following statements.
(a) There is a function $f : A \to A$ so that $f \circ f = f$ and $f \neq i_A$.
Solution: The statement is true. Consider the function $f : A \to A$ defined by $f(x) = 1$ for all $x \in A$. Then for any $x \in A$, $f \circ f(x) = f(f(x)) = f(1) = 1 = f(x)$. Thus, $f \circ f = f$. However, $f \neq i_A$ because $f(2) = 1 \neq 2 = i_A(2)$.

(b) There is a function $g : A \to A$ so that $g \circ g = g$ and g is one-to-one, but $g(3) \neq 3$.
Solution: The statement is false. We prove that there can not be a function $g : A \to A$ so that $g \circ g = g$ and g is one-to-one, but $g(3) \neq 3$ by a contradiction proof. Suppose that there is a $g : A \to A$ so that $g \circ g = g$ and g is one-to-one, but $g(3) \neq 3$. Then since $g \circ g = g$, we get $g(g(3)) = g \circ g(3) = g(3)$. Since $g(g(3)) = g(3)$ and g is one-to-one, we get that $g(3) = 3$, which contradicts the fact that $g(3) \neq 3$.

(c) There is a function $h : A \to A$ so that $h \circ h = i_A$ and $h \neq i_A$.
Solution: The statement is true. Consider the function $h : A \to A$ defined by $h(1) = 2$, $h(2) = 1$, $h(3) = 3$, and $h(4) = 4$. Then

$$\begin{aligned} h \circ h(1) &= h(h(1)) = h(2) = 1 = i_A(1) \\ h \circ h(2) &= h(h(2)) = h(1) = 2 = i_A(2) \\ h \circ h(3) &= h(h(3)) = h(3) = 3 = i_A(3) \\ h \circ h(4) &= h(h(4)) = h(4) = 4 = i_A(4) \end{aligned}$$

Thus, $h \circ h = i_A$, but $h \neq i_A$ because $h(1) = 2 \neq 1 = i_A(1)$.