MATHEMATICS 271 L01 FALL 2004
QUIZ 2 SOLUTION

1. Prove or disprove the following statements:
(a) For real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.
Solution: This statement is false because when $x = -0.5$ we have $\lfloor x^2 \rfloor = \left\lfloor (-0.5)^2 \right\rfloor = \lfloor 0.25 \rfloor = 0 \neq 1 = (-1)^2 = \lfloor -0.5 \rfloor^2$.
(b) For real numbers x, if $x - \lfloor x \rfloor < \frac{1}{2}$ then $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$.

Solution: This statement is true and here is a proof. Let x be a real number so that $x - \lfloor x \rfloor < \frac{1}{2}$. We show that $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$. From $x - \lfloor x \rfloor < \frac{1}{2}$, we get that

$2x < 2 \lfloor x \rfloor + 1$ (1)

Since $\lfloor x \rfloor \leq x$, we see that

$2 \lfloor x \rfloor \leq 2x$ (2)

From (1) and (2), we see that $2 \lfloor x \rfloor$ is an integer satisfying the condition $2 \lfloor x \rfloor \leq 2x < 2 \lfloor x \rfloor + 1$, which means $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$.

2. Prove or disprove the following statements:
(a) For all non-negative integers n, $n^2 - n + 11$.
Solution: This statement is false because when $n = 11$ we have $n^2 - n + 11 = (11)^2 - 11 + 11 = (11)^2$ which is not a prime.

(b) For any integers n, $n(n^2 - 1)(n + 2)$ is divisible by 4.
Solution: This statement is true and here is a proof. Let n be an integer. We have two cases.
Case 1: n is even, that is, $n = 2k$ for some integer k. Then

$n(n^2 - 1)(n + 2) = 2k(n^2 - 1)(2k + 2) = 4k(n^2 - 1)(k + 1)$

which implies that $n(n^2 - 1)(n + 2)$ is divisible by 4 (note that $k(n^2 - 1)(k + 1)$ is an integer).
Case 2: n is even, that is, $n = 2k + 1$ for some integer k. Then

$n(n^2 - 1)(n + 2) = n\left((2k + 1)^2 - 1\right)(n + 2) = n(4k^2 + 4k + 1 - 1)(n + 2) = n(4k^2 + 4k)(n + 2) = 4n(k^2 + k)(n + 2)$

which implies that $n(n^2 - 1)(n + 2)$ is divisible by 4 (note that $n(k^2 + k)(n + 2)$ is an integer).