MATHEMATICS 271 L01 FALL 2006
ASSIGNMENT 1 SOLUTION

1. Let $\mathcal{P}$ be the statement : "For all real numbers x and y, if $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$ then $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$."

(a) Is $\mathcal{P}$ true? Prove your answer.
Solution: $\mathcal{P}$ is true and here is a proof. Let x and y be real numbers and suppose that $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$. It follows that $x < \lfloor x \rfloor + \frac{1}{2}$ and $y < \lfloor y \rfloor + \frac{1}{2}$, and so,

$$x + y < \left(\lfloor x \rfloor + \tfrac{1}{2}\right) + \lfloor y \rfloor + \tfrac{1}{2} = \lfloor x \rfloor + \lfloor y \rfloor + 1. \qquad (1)$$

Next, from the definition of the floor, we have $\lfloor x \rfloor \leq x$ and $\lfloor y \rfloor \leq y$, and so

$$\lfloor x \rfloor + \lfloor y \rfloor \leq x + y \qquad (2).$$

From (1) and (2), we get, $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y < \lfloor x \rfloor + \lfloor y \rfloor + 1$ where $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer, and so $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

(b) State the *converse* of $\mathcal{P}$. Is the *converse* of $\mathcal{P}$ true? Prove your answer.
Solution: The *converse* of $\mathcal{P}$ is "For all real numbers x and y, if $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ then $x - \lfloor x \rfloor < \frac{1}{2}$ and $y - \lfloor y \rfloor < \frac{1}{2}$."

The converse of is false. For example, when $x = 0.7$ and $y = 0$, we have $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, but $x - \lfloor x \rfloor = 0.7 - 0 = 0.7 \geq \frac{1}{2}$.

(c) State the the *contrapositive* of $\mathcal{P}$. Is the *contrapositive* of $\mathcal{P}$ true? Explain.
Solution: The contrapositive of $\mathcal{P}$ is "For all real numbers x and y, if $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$ then $x - \lfloor x \rfloor \geq \frac{1}{2}$ or $y - \lfloor y \rfloor \geq \frac{1}{2}$."

(d) State the *negation* of $\mathcal{P}$. Is the *negation* of $\mathcal{P}$ true? Explain
Solution: The negation of $\mathcal{P}$ is "There exist real numbers x and y so that if $x - \lfloor x \rfloor < \frac{1}{2}$ and. $y - \lfloor y \rfloor < \frac{1}{2}$ but $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$."

2. Prove or disprove each of the following statement..

(a) For all integers a, b and c, if $a \mid b$ and $a \mid c$ then $a \mid xb + yc$ for all integers x and y.
Solution: This statement is true and here is a proof. Let a, b and c be integers and suppose that $a \mid b$ and $a \mid c$. Let x and y be any integers. Since $a \mid b$ and $a \mid c$, there are integers m and n so that $b = am$ and $c = an$. It follows that $xb + yc = xam = yam = a\,(xm + yn)$ where $xm + yn$ is an integer. Thus, $a \mid xb + yc$.

(b) For all integers a, b and c, $a \mid b$ and $a \mid c$ if and only if $a \mid 2b + c$ and $a \mid b + 2c$.
Solution: This statement is false. For example, when $a = 3$, and $b = c = 1$, we see that $a \mid 2b + c$ and $a \mid b + 2c$, but $a \nmid b$.

(c) For all integers a, b and c, $a \mid b$ and $a \mid c$ if and only if $a \mid 2b + 3c$ and $a \mid b + 2c$.
Let a, b and c be integers.

First, we prove that if $a \mid b$ and $a \mid c$ then $a \mid 2b+3c$ and $a \mid b+2c$. This statement is true by part (a).
Next we prove that if $a \mid 2b+3c$ and $a \mid b+2c$ then $a \mid b$ and $a \mid c$. Suppose that $a \mid 2b+3c$ and $a \mid b+2c$, that is, $2b+3c = am$ and $b+2c = an$ for some integers m and n. Then $b = 2(2b+3c) - 3(b+2c) = 2am - 3an = a(2m-3n)$ and $c = 2(b+2c) - (2b+3c) = 2an - am = a(2n-m)$ where $2m-3n$ and $2n-m$ are integers. It folllows that $a \mid 2b+3c$ and $a \mid b+2c$.

3. For each of the following statements, determine whether the statement is true or false and prove your answer.

(a) For all integers x, x^3+x is even.
Solution: This statement is true and here is a proof. Let x be an integer. We have two cases.
Case 1: x is even. Then there is an integer k so that $x = 2k$, and so $x^3+x = x(x^2+1) = 2k(x^2+1)$ where $k(x^2+1)$ is an integer, which implies that x^3+x is even.
Case 2: x is odd. Then there is an integer m so that $x = 2m+1$, and so $x^3+x = x(x^2+1) = x\left((2m+1)^2+1\right) = x(4m^2+4m+2) = 2x(2m^2+2m+1)$ where $x(2m^2+2m+1)$ is an integer, which implies that x^3+x is even.

(b) For all integers y, there is an integer x so that $x^3+x = y$.
Solution: This statement is false For example, let $y=1$. Then for any integers x, by part (a), x^3+x is even, so $x^3+x \neq 1 = y$.

(c) For all integers x and y, if $x^3+x = y^3+y$ then $x = y$.
Solution: This statement is true and here is a proof. Let x, y be integer so that $x^3+x = y^3+y$. Then $(x^3+x) - (y^3+y) = 0$, which can be simplified as
$(x-y)() = 0$ (1).

Now, $x^2+xy+y^2+1 = x^2+2\left(\frac{y}{2}\right)x + \left(\frac{y}{2}\right)^2 + \frac{y^2}{4} + 1 = \left(x+\frac{y}{2}\right)^2 + \frac{y^2}{4} + 1 \geq 1$, so $x^2+xy+y^2+1 \neq 0$ and hence, from (1) we get $x-y=0$, and so $x=y$.