MATHEMATICS 271 L01 FALL 2007
ASSIGNMENT 2 SOLUTION

1. Prove or disprove the following statements:

(a) For real numbers x and y, $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$.

For example, when $x = y = 0.1$, $\lceil x+y \rceil = \lceil 0.2 \rceil = 1 \neq 2 = \lceil x \rceil + \lceil y \rceil$.

(b) For real numbers x and y, if $x + \lceil x \rceil = y + \lceil y \rceil$ then $x = y$.

Solution: This statement is true. Let x and y be real numbers such that

$$x + \lceil x \rceil = y + \lceil y \rceil \qquad (1)$$

. We note that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ and $\lceil y \rceil - 1 < y \leq \lceil y \rceil$, and so $0 \leq \lceil x \rceil - x < 1$ and $0 \leq \lceil y \rceil - y < 1$. Let $x' = \lceil x \rceil - x$ and $y' = \lceil y \rceil - y$. We see that $x = \lceil x \rceil - x'$ and $y = \lceil y \rceil - y'$ where $0 \leq x', y' < 1$, and (1) becomes $\lceil x \rceil - x' + \lceil x \rceil = \lceil y \rceil - y' + \lceil y \rceil$ and therfore,

$$2\left(\lceil x \rceil - \lceil y \rceil\right) = x' - y' \qquad (2)$$

Since $0 \leq x', y' < 1$, we see that $-1 < x' - y' < 1$ (3)

From (2) and (3), we see that $x' - y'$ is an integer strictly between -1 and 1, and thus $x' - y' = 0$, and so from (2), we get $\lceil x \rceil = \lceil y \rceil$. It follows that $x = \lceil x \rceil - x' = \lceil y \rceil - y' = y$.

(c) For real numbers y, there is a real number x such that $y = x + \lceil x \rceil$.

Solution: This statement is false. We can prove that for all real numbers x, $x + \lceil x \rceil \neq 1$ by a contradiction proof. Suppose that there exists a real number x such that $x + \lceil x \rceil = 1$. Then $x = 1 - \lceil x \rceil$ which is an integer and so $\lceil x \rceil = x$ and therefore we have $2\lceil x \rceil = 1$ which implies that $\lceil x \rceil = \frac{1}{2}$ which is not an integer. Thus, for all real numbers x, $x + \lceil x \rceil \neq 1$.

2. The Fibonacci sequence f_1, f_2, $f_3 \cdots$ is defined by $f_1 = f_2 = 1$ and for integers $k \geq 3$, $f_k = f_{k-1} + f_{k-2}$.

(a) Prove that for all integers $n \geq 2$.

Solution: We prove that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$ by induction on n.

Base Case: $(n = 2, 3)$

$f_2 = 1 < \frac{7}{4} = \left(\frac{7}{4}\right)^{2-1}$ and $f_3 = f_2 + f_1 = 1 + 1 = 2 = \frac{32}{16} < \frac{49}{16} = \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{3-1}$.

Inductive step: Let $k \geq 4$ be an integer and suppose that

$f_m < \left(\frac{7}{4}\right)^{m-1}$ for all integers m where $2 \leq m < k$. (IH)

We want to prove that $f_k < \left(\frac{7}{4}\right)^{k-1}$.

Now,

$$\begin{array}{rcll} f_k & = & f_{k-1} + f_{k-2} & \text{because } k \geq 4 \\ & < & \left(\frac{7}{4}\right)^{k-2} + \left(\frac{7}{4}\right)^{k-3} & \text{by (IH) and because} 2 \leq k-1, k-2 < k. \\ & = & \left(\frac{7}{4}\right)^{k-1}\left(\frac{4}{7} + \frac{16}{49}\right) & \\ & = & \left(\frac{7}{4}\right)^{k-1}\left(\frac{28+16}{49}\right) & \\ & = & \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{49}\right) & \\ & < & \left(\frac{7}{4}\right)^{k-1} & \text{because } \frac{44}{49} < 1. \end{array}$$

Thus, by the Principle of Mathematical Induction (Strong Form), we conclude that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

(b) Prove that $\sum_{i=1}^{n} {f_i}^2 = f_{n+1}f_n$ for all integers $n \geq 1$.

Solution: We prove that for all integers $n \geq 1$ by induction on n.

Base Case: $(n = 1)$

$\sum_{i=1}^{1} {f_i}^2 = f_1^2 = 1^2 = 1 = 1 \times 1 = f_2 f_1 = f_{1+1}f_1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that

$\sum_{i=1}^{k} {f_i}^2 = f_{k+1}f_k$. (IH)

We want to prove that $\sum_{i=1}^{k+1} {f_i}^2 = f_{k+2}f_{k+1}$

Now,

$$\begin{array}{rcll} \sum_{i=1}^{k+1} {f_i}^2 & = & \left(\sum_{i=1}^{k} {f_i}^2\right) + f_{k+1}^2 & \\ & = & f_{k+1}f_k + f_{k+1}^2 & \text{by (IH).} \\ & = & (f_k + f_{k+1})\, f_{k+1} & \\ & = & f_{k+2}f_{k+1} & \text{because } f_{k+2} = f_k + f_{k+1} \end{array}$$

Thus, by the Principle of Mathematical Induction, we conclude that $\sum_{i=1}^{n} {f_i}^2 = f_{n+1}f_n$ for all integers $n \geq 1$.

(c) Prove that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all integers $n \geq 1$.

Solution: We prove that for all integers $n \geq 1$ by induction on n.

Base Case: $(n = 1)$

$\sum_{i=1}^{1} f_i = f_1 = 1 = 2 - 1 = f_3 - 1 = f_{1+2} - 1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that

$\sum_{i=1}^{k} f_i = f_{k+2} - 1$. (IH)

We want to prove that $\sum_{i=1}^{k+1} f_i = f_{k+3} - 1$.

Now,

$$\begin{aligned}
\sum_{i=1}^{k+1} f_i &= \left(\sum_{i=1}^{k} f_i\right) + f_{k+1} \\
&= f_{k+2} - 1 + f_{k+1} && \text{by (IH).} \\
&= f_{k+2} + f_{k+1} - 1 \\
&= f_{k+3} - 1 && \text{because } f_{k+3} = f_{k+2} + f_{k+1}
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, we conclude that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all integers $n \geq 1$.

3. Prove the following statements by induction on n:

(a) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2\left(\sqrt{n+1} - 1\right)$ for all integers $n \geq 1$.

Solution: We prove that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2\left(\sqrt{n+1} - 1\right)$ for all integers $n \geq 1$ by induction on n.

Base Case: $(n = 1)$

Since $1^2 = 1 < 2 < 2.25 = \left(\frac{3}{2}\right)^2$, we see that $1 < \sqrt{2} < \frac{3}{2}$, and hence $\sqrt{2} - 1 < \frac{1}{2}$. Thus, $\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1 = 2 \times \frac{1}{2} > 2\left(\sqrt{2} - 1\right) = 2\sqrt{1+1} - 1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > 2\left(\sqrt{k+1} - 1\right). \qquad \text{(IH)}$$

We want to prove that $\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > 2\left(\sqrt{k+2} - 1\right)$.

Now,

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \left(\sum_{i=1}^{k} \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{k+1}} \\
&> 2\left(\sqrt{k+1} - 1\right) + \frac{1}{\sqrt{k+1}} && \text{by (IH).} \\
&= 2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} - 2 \\
&= 2\left(\sqrt{k+1} + \frac{1}{2\sqrt{k+1}}\right) - 2 \\
&= 2\left(\frac{2(k+1)+1}{2\sqrt{k+1}}\right) - 2 \\
&= 2\left(\frac{2k+3}{\sqrt{4(k+1)}}\right) - 2 \\
&= 2\left(\frac{(2k+3)}{\sqrt{4(k+1)(k+2)}}\sqrt{k+2}\right) - 2 \\
&= 2\left(\sqrt{\frac{(2k+3)^2}{4(k+1)(k+2)}}\sqrt{k+2}\right) - 2 \\
&= 2\left(\sqrt{\frac{4k^2+12k+9}{4k^2+12k+8}}\sqrt{k+2}\right) - 2 \\
&> 2\sqrt{k+2} - 2 && \text{because } \frac{4k^2+12k+9}{4k^2+12k+8} > 1 \\
&= 2\left(\sqrt{k+2} - 1\right).
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, we conclude that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > 2\left(\sqrt{n+1} - 1\right)$ for all integers $n \geq 1$.

(b) $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \geq \dfrac{1}{2n}$ for all integers $n \geq 1$.

Solution: We prove that $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \geq \dfrac{1}{2n}$ for all integers $n \geq 1$ by induction on n.

Base Case: $(n = 1)$

$\dfrac{1}{2} \geq \dfrac{1}{2} = \dfrac{1}{2\times 1}$

Inductive step: Let $k \geq 1$ be an integer and suppose that

$$\frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot(2k-1)}{2\cdot 4\cdot 6\cdots(2k-2)\cdot(2k)} \geq \frac{1}{2k} \qquad (IH)$$

We want to show that

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2k+1)}{2\cdot 4\cdot 6\cdots(2k)\cdot(2k+2)} \geq \frac{1}{2k+2}.$$

Now,

$$\begin{aligned}\frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2k+1)}{2\cdot 4\cdot 6\cdots(2k)\cdot(2k+2)} &= \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)} \times \frac{(2k+1)}{(2k+2)} \\ &\geq \frac{1}{2k}\times\frac{(2k+1)}{(2k+2)} && \text{by } (IH) \\ &= \frac{2k+1}{2k}\times\frac{1}{2k+2} \\ &\geq \frac{1}{2k+2} && \text{because } \frac{2k+1}{2k} \geq 1\end{aligned}$$

Thus, by the Principle of Mathematical Induction, we conclude that $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \geq \dfrac{1}{2n}$ for all integers $n \geq 1$.

(c) $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \leq \dfrac{1}{\sqrt{n+1}}$ for all integers $n \geq 1$.

Solution: We prove that $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \leq \dfrac{1}{\sqrt{n+1}}$ for all integers $n \geq 1$ by induction on n.

Base Case: $(n = 1)$

$\dfrac{1}{2} \leq \dfrac{1}{2} = \dfrac{1}{\sqrt{1+1}}$

Inductive step: Let $k \geq 1$ be an integer and suppose that

$$\frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot(2k-1)}{2\cdot 4\cdot 6\cdots(2k-2)\cdot(2k)} \leq \frac{1}{\sqrt{k+1}} \qquad (IH).$$

We want to show that

$$\frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2k+1)}{2\cdot 4\cdot 6\cdots(2k)\cdot(2k+2)} \leq \frac{1}{\sqrt{k+2}}.$$

Now,

$$\begin{aligned}
\frac{1\cdot 3\cdot 5\cdots(2k-1)\cdot(2k+1)}{2\cdot 4\cdot 6\cdots(2k)\cdot(2k+2)} &= \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\times\frac{2k+1}{2k+2} \\
&\leq \frac{1}{\sqrt{k+1}}\times\frac{2k+1}{2k+2} && \text{by } (IH) \\
&= \frac{1}{\sqrt{k+1}}\times\frac{2k+1}{2k+2}\times\frac{\sqrt{k+2}}{\sqrt{k+2}} \\
&= \sqrt{\frac{(2k+1)^2(k+2)}{(2k+2)^2(k+1)}}\times\frac{1}{\sqrt{k+2}} \\
&= \sqrt{\frac{(4k^2+4k+1)(k+2)}{(4k^2+8k+4)(k+1)}}\times\frac{1}{\sqrt{k+2}} \\
&= \sqrt{\frac{4k^3+12k^2+9k+2}{4k^3+12k^2+12k+4}}\times\frac{1}{\sqrt{k+2}} \\
&\leq \frac{1}{\sqrt{k+2}} && \text{because } \frac{4k^3+12k^2+9k+2}{4k^3+12k^2+12k+4}
\end{aligned}$$

Thus, by the Principle of Mathematical Induction, we conclude that $\frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} \leq \frac{1}{\sqrt{n+1}}$ for all integers $n \geq 1$.