MATHEMATICS 271 L01 FALL 2007
ASSIGNMENT 3 SOLUTION

1. Prove or disprove each of the following statements:

(a) For all sets A and B, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution: This statement is false. Let $A = \{1\}, B = \{2\}$. Then $\{1,2\} \in \mathcal{P}(A \cup B)$, but $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ because $\{1,2\} \nsubseteq A$ and $\{1,2\} \nsubseteq B$.

(b) For all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Solution: This statement is true. Let A, B be sets. Let $X \in \cup\mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(A)$.

Case 1: $X \in \mathcal{P}(A)$, then $X \subseteq A$ and so $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

Case 2: $X \in \mathcal{P}(B)$, then $X \subseteq B$ and so $X \subseteq A \cup B$, hence $X \in \mathcal{P}(A \cup B)$.

(c) For all sets A, B and C, $(A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$.

Solution: This statement is false. For example, when $A = D = \{1\}$ and $B = C = \emptyset$, $(A \cup B) \times (C \cup D) = \{(1,1)\} \neq \emptyset = (A \times C) \cup (B \times D)$

(d) For all sets A, B and C, if $A \subseteq (B - C)$ then $A \cap (C - B) = \emptyset$.

Solution: This statement is true. Let A, B and C be sets and suppose that $A \subseteq (B - C)$. We prove that $A \cap (C - B) = \emptyset$ by a contadiction proof. Suppose that $A \cap (C - B) \neq \emptyset$, that is, there exists an element $x \in A \cap (C - B)$. Since $x \in A \cap (C - B)$, $x \in A$ and $x \in C - B$. Since $x \in C - B$, $x \in C$ and hence $x \notin B - C$. Thus, there exists an element x such that $x \in A$ and $x \notin B - C$, which implies that $A \nsubseteq (B - C)$. This contradicts the assumption that $A \subseteq (B - C)$. Thus, $A \cap (C - B) = \emptyset$.

2. For any sets A and B, we define the *symmetric difference* $A \triangle B$ by $A \triangle B = (A \cup B) - (A \cap B)$. Note that it is also true that $A \triangle B = (A - B) \cup (B - A)$. Let $\mathcal{S}$ be the statement: "*For all sets A, B and C, if $A \subseteq B \cup C$ and $B \subseteq C \cup A$ then $A \triangle B = C$.*" and let $\mathcal{T}$ be the statement:"*For all sets A, B and C, if $A \triangle B = A \triangle C$ then $B \subseteq C$.*"

(a) Is $\mathcal{S}$ true? Prove your answer.

Solution: $\mathcal{S}$ is not true. For example, when $A = B = \emptyset$, and $C = \{1\}$ we have $A = \emptyset \subseteq B \cup C$ and $B = \emptyset \subseteq C \cup A$, but $A \triangle B = \emptyset \neq \{1\} = C$.

(b) Is $\mathcal{T}$ true? Prove your answer.

Solution: $\mathcal{T}$ true. Let A, B and C be sets and suppose that $A \triangle B = A \triangle C$. We prove that $B \subseteq C$. Let $x \in B$. We consider two cases $x \in A$ and $x \notin A$.

Case 1: $x \in A$. Since $x \in A$ and $x \in B$, $x \in A \cap B$ and hence $x \notin A \triangle B$. Since $A \triangle B = A \triangle C$ and $x \notin A \triangle B$, we get $x \notin A \triangle C = (A \cup C) - (A \cap C)$ and therefore $x \notin A \cup C$ or $x \in A \cap C$. However, since $x \in A$, $x \in A \cup C$ and it follows that $x \in A \cap C$ and so $x \in C$.

Case 1: $x \notin A$. Since $x \notin A$ and $x \in B$, $x \in B - A$ and hence $x \in A \triangle B$. Since $A \triangle B = A \triangle C$ and $x \in A \triangle B$, we get $x \in A \triangle C = (A \cup C) - (A \cap C)$ and therefore $x \in A \cup C$, which implies that $x \in A$ or $x \in C$. However, since $x \notin A$, we see that $x \in C$.

(c) Write the converse of $\mathcal{S}$. Is the converse of $\mathcal{S}$ true? Prove your answer.

Solution: The converse of $\mathcal{S}$ is "*For all sets A, B and C, if $A \triangle B = C$ then $A \subseteq B \cup C$ and $B \subseteq C \cup A$.*"

The converse of $\mathcal{S}$ is true. Let A, B and C be sets and suppose that $A \triangle B = C$.

First, we prove that $A \subseteq B \cup C$ by contradiction. Suppose that $A \nsubseteq B \cup C$. Then there exists an element $x \in A$ so that $x \notin B \cup C$, that is, $x \notin B$ and $x \notin C$. Since $x \in A$ and $x \notin B$, $x \in A - B \subseteq A \triangle B$. Since we have $x \in A \triangle B$ and $x \notin C$, $A \triangle B \neq C$ which contradicts the assumption that $A \triangle B = C$. Thus, $A \subseteq B \cup C$.

Next, we prove that $B \subseteq C \cup A$ by contradiction. Suppose that $B \nsubseteq C \cup A$. Then there exists an element $x \in B$ so that $x \notin C \cup A$, that is, $x \notin C$ and $x \notin A$. Since $x \in B$ and $x \notin A$, $x \in B - A \subseteq A \triangle B$. Since we have $x \in A \triangle B$ and $x \notin C$, $A \triangle B \neq C$ which contradicts the assumption that $A \triangle B = C$. Thus, $A \subseteq B \cup C$.

3. Let $S = \{1000, 1001, 1002, ..., 9999\}$. For each of the following questions, you must explain how you got the answer.

(a) How many numbers in S have at least one digit that is a 2 or a 5?

Solution: The answer to this question is $9 \times 10^3 - 7 \times 8^3 = 5416$, which is the number of numbers in S minus the number of numbers in S which do not have a 2 nor a 5 (for there are 7 choices (1,3,4,6,7,8, or 9) for the first digit and there are 8 choices (0,1,3,4,6,7,8, or 9) for each of the 3 remaining digits).

(b) How many numbers in S have at least one digit that is a 2 and at least one digit that is a 5?

Solution: Let A be the set of numbers in S which have no 2's and let B be the set of numbers in S which have no 5's. Then, with the reasons similar to part (a), we see that $|A| = |B| = 8 \times 9^3$ and $|A \cap B| = 7 \times 8^3$. The answer to part (b) is

$$|S| - |A \cup B| = |S| - (|A| + |B| - |A \cap B|) = 9 \times 10^3 - (2 \times 8 \times 9^3 - 7 \times 8^3) = 920.$$

(c) How many numbers in S have the property that the sum of its digits is even?

Solution: Since the sum of the digits is even, the number of odd digits is even, and so we have three mutually disjoint cases: (1) there are no odd digits, (2) there are 2 odd digits and (3) all the digits are odd. We have 4×5^3 numbers in S in which there are no odd digits (there are 4 choices (2, 4, 6 or 8) for the first digit, and there are 5 choices (0, 2, 4, 6 or 8) for each of the 3 remaining digits). Similarly, there are 5^4 numbers in S in which all the digits are odd (there are 5 choices (1, 3, 5, 7 or 9 for each of the 4 digits). To count the numbers in S which have exactly 2 odd digits, we divide into two cases: the first digit is odd or the first digit is even. The number of numbers in S which have exactly 2 odd digits and the first digit is odd is $5 \times 3 \times 5^3$ (5 choices for the first digit, 3 choices for where the second odd digit is, and 5 choices for each of the remaining 3 digits). The number of numbers in S which have exactly 2 odd digits and the first digit is even is $4 \times 3 \times 5^3$ (4 choices for the first digit, 3 choices for where the second even digit is, and 5 choices for each of the remaining 3 digits). The answer to (c) is $4 \times 5^3 + 5^4 + 5 \times 3 \times 5^3 + 4 \times 3 \times 5^3 = 4500$.

(d) How many numbers in S have the property that the digits appear in increasing order (that is, the first digit is smaller than the second digit, the second digit is smaller than the third digit, and the third digit is smaller than the fourth digit)?

Solution: The answer to this question is $\binom{9}{4}$, that is, we choose 4 digits from 1 to 9 and then arrange them in the increasing order.