MATHEMATICS 271 L01 FALL 2007
ASSIGNMENT 4 SOLUTION

1. Let $h : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $h(x) = 2x + 1$ for all $x \in \mathbb{Z}$.

(a) Prove that h is one-to-one but not onto $\mathbb{Z}$.

Solution: First, we prove that h is one-to-one. Let $x, y \in \mathbb{Z}$ so that $h(x) = h(y)$, that is $2x + 1 = 2y + 1$. Then, $x = \frac{1}{2}(2x + 1 - 1) = \frac{1}{2}(2y + 1 - 1) = y$.

Next, we prove that h is not onto $\mathbb{Z}$. Since for all $x \in \mathbb{Z}$, $h(x) = 2x + 1$ which is odd, we see that $h(x) \neq 0$ for all $x \in \mathbb{Z}$, and so h is not onto $\mathbb{Z}$.

(b) Is

Solution: Yes, there is a function $g : \mathbb{Z} \to \mathbb{Z}$ so that $g \circ h$ is onto $\mathbb{Z}$. For example, let $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = \left\lfloor \frac{x-1}{2} \right\rfloor$ for any $x \in \mathbb{Z}$. Then for any $x \in \mathbb{Z}$, $g \circ h(x) = g(h(x)) = \left\lfloor \frac{h(x)-1}{2} \right\rfloor = \left\lfloor \frac{2x+1-1}{2} \right\rfloor = \lfloor x \rfloor = x$, and so $g \circ h$ is onto $\mathbb{Z}$.

(c) For positive integers n, define the functions $h^n : \mathbb{Z} \to \mathbb{Z}$ by putting $h^1 = h$ and for integers $k \geq 2$, $h^k = h \circ h^{k-1}$. Compute $h^2(x)$, $h^3(x)$, $h^4(x)$, and $h^5(x)$. From these, guess an explicit formula for $h^n(x)$ for all integers $n \geq 1$.

Solution:

$$\begin{array}{lllllllll} h^2(x) & = & h(h(x)) & = & 2h(x)+1 & = & 2(2x+1)+1 & = & 4x+3 \\ h^3(x) & = & h(h^2(x)) & = & 2h^2(x)+1 & = & 2(4x+3)+1 & = & 8x+7 \\ h^4(x) & = & h(h^3(x)) & = & 2h^3(x)+1 & = & 2(8x+7)+1 & = & 16x+15 \\ h^5(x) & = & h(h^4(x)) & = & 2h^4(x)+1 & = & 2(16x+15)+1 & = & 32x+31 \end{array}$$

From the above, we guess that $h^n(x) = 2^n x + 2^n - 1$ for all integers $n \geq 1$.

(d) Prove by induction on n that your guess in part (c) is correct.

Solution:

Base case: $(n = 1)$

$h^1(x) = h(x) = 2x + 1 = 2^1 x + 2^1 - 1$.

Inductive step: Let $k \geq 1$ be an integer and suppose that

$$h^k(x) = 2^k x + 2^k - 1. \qquad (IH)$$

We want to show that $h^{k+1}(x) = 2^{k+1}x + 2^{k+1} - 1$.

Now,

$$\begin{array}{lll} h^{k+1}(x) & = & h\left(h^k(x)\right) \\ & = & 2h^k(x) + 1 \\ & = & 2\left(2^k x + 2^k - 1\right) + 1 \\ & = & 2^{k+1}x + 2^{k+1} - 2 + 1 \\ & = & 2^{k+1}x + 2^{k+1} - 1. \end{array}$$

Thus, by the Principle of Mathematical Induction, we conclude that $h^n(x) = 2^n x + 2^n - 1$ for all integers $n \geq 1$.

2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove or disprove the following statements:

(a) If both f and g are onto then $g \circ f$ is onto.

Solution:

This statement is true. Let $f : X \to Y$ and $g : Y \to Z$ be onto functions. We prove that $g \circ f$ is onto. Let $z \in Z$. Since $z \in Z$ and g is onto Z, there is $y \in Y$ so that $g(y) = z$. Since $y \in Y$ and f is onto Y, there is $x \in X$ so that $f(x) = y$. Thus, there is $x \in X$ so that $g \circ f(x) = g(f(x)) = g(y) = z$. Thus, $g \circ f$ is onto.

(b) If $g \circ f$ is onto then f is onto.

Solution:

This statement is false. For example, when $X = Z = \{1\}$, $Y = \{1, 2\}$, $f = \{(1, 1)\}$, $g = \{(1, 1), (2, 1)\}$, we see that $g \circ f$ is onto, but f is not onto.

(c) If $g \circ f$ is onto then g is onto.

Solution:

Let $z \in Z$. Since $g \circ f$ is onto, there exist $x \in X$ so that $g \circ f(x) = z$. Now, let $y = f(x) \in Y$. Then $g(y) = g(f(x)) = g \circ f(x) = z$. Thus, g is onto.

(d) If $g \circ f$ is onto and g is one-to-one then f is onto.

Solution:

This statement is true. Suppose that $g \circ f$ is onto and g is one-to-one. We prove that f is onto. Let $y \in Y$. Since $g(y) \in Z$ and $g \circ f$ is onto, there is $x \in X$ so that $g \circ f(x) = g(y)$. Since $g(f(x)) = g \circ f(x) = g(y)$ and g is one-to-one, we get $f(x) = y$. Thus, f is onto.

3. Let $A = \{1, 2, 3, 4\}$. You must provide a proof, or an explanation to support your answer to each of the following questions.

(a) Is it true that if $h : A \to A$ is a function so that $h \circ h(1) = 2$ then h must be onto?

Solution: No, it is not true. For example, let $h : A \to A$ defined by $h(x) = 2$ for all $x \in A$. Then $h \circ h(1) = 2$ but h is not onto.

(b) How many functions $f : A \to A$ are there so that $f \circ f(1) = 2$?

Solution:

The answer is $3 \times 1 \times 4 \times 4 = 48$ because there are three choices for $f(1)$ namely 2, 3 or 4 (note that $f(1) \neq 1$, for if $f(1) = 1$ then $f \circ f(1) = f(f(1)) = f(1) = 1 \neq 2$), there are 1 choice for the image of $f(1)$ under f namely 2, and there are 4 choices for the images of each of the two remaining elements of A under f .

(c) How many onto functions $f : A \to A$ are there so that $f \circ f(1) = 2$?

Solution: We note that since A is finite, $f : A \to A$ is onto if and only if it is one-to-one. The answer is $2 \times 1 \times 2 \times 1 = 4$ because there are 2 choices for $f(1)$ namely 3 or 4 (we know that $f(1) \neq 1$, as noted above, and also $f(1) \neq 2$ for if $f(1) = 2$ then the condition $f \circ f(1) = 2$ becomes $f(2) = 2$ and so f is not one-to-one), there are 1 choice for the image of $f(1)$ under f namely 2, and there are 2 choices for the images of first remaining element of A under f and there is 1 choice for the remaining element of A under f .

(d) Is it true that if f, g, h are a functions from A to A so that $f \circ h = g \circ h$ then $f = g$?

Solution:

No it is not true. For example, when f, g, h are a functions from A to A are defined by $f(x) = h(x) = 2$ for all $x \in A$, and $g(x) = 1$ whenever $x \neq 2$ and $g(2) = 2$. Then for any $x \in A$, $f \circ h(x) = f(h(x)) = f(2) = g(2) = g(h(x)) = g \circ h(x)$, and hence $f \circ h = g \circ h$. However, $f \neq g$ because $f(1) = 2 \neq 1 = g(1)$.

(e) Is there a

Solution:

No, there is no such function. We prove that by contradiction. Suppose that there is a function $f : A \to A$ so that $f \circ f = f$, f is onto and $f \neq i_A$. As noted above, f is also one-to-one and so f is invertible, that is f^{-1} exists. Then $f = f \circ i_A = f \circ (f \circ f^{-1}) = (f \circ f) \circ f^{-1} = f \circ f^{-1} = i_A$ which contradicts the assumption that $f \neq i_A$.