MATHEMATICS 271 L20 SPRING 2005
ASSIGNMENT 2

This assignment is to be handed in on Wednesday, June 15, 2005 at 7:00 p.m.. Late assignments will not be accepted and are given a mark of zero. Students should attempt all problems. However, only one problem will be marked for credit.

1.

(a) Prove by a combinatorial proof that for all positive integers $n \geq k \geq m$, $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

(b) Prove by a combinatorial proof that for all positive integers w and m, $\binom{w+m}{2} = \binom{w}{2} + wm + \binom{m}{2}$.

(c) Let k be a fixed positive integer. Prove by induction on n that $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ for all integers $n \geq k$.

(d) Prove that for all integers n and k, where $n \geq k+2$ and $k \geq 2$, $\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2}$ is even.

2. An urn contains ten white balls numbered from 1 to 10, and ten black balls numbered from 1 to 10. A sample of 5 balls is chosen from the urn.

(a) How many different samples are there?

(b) How many samples in (a) have at least one white ball?

(c) How many samples in (a) have the property that the sum of the numbers on the balls is even?

(d) How many samples in (a) have the property that the product of the numbers on the balls is even?

(e) How many samples in (a) have the property that the sum of the numbers on the balls is odd but the product of the numbers on the balls is even?

(f) How many samples in (a) have the property that the numbers on the balls are distinct?

3. Prove or disprove each of the following statements.

(a) For all sets A, B, C and D, $(A \times B) - (C \times D) \subseteq (A - C) \times (B - D)$.

(b) For all sets A, B, C and D, $(A - C) \times (B - D) \subseteq (A \times B) - (C \times D)$.

(c) For all sets A, B, and C, if $A \triangle B = A \triangle C$ then $B = C$.

(d) For all sets A and B, $P(A \cup B) = P(A) \cup P(B)$.

(e) For all sets A and B, $P(A \cap B) = P(A) \cap P(B)$.

4. The Fibonacci sequence $f_1, f_2, f_3 \cdots$ is defined by $f_1 = f_2 = 1$ and for integers $k \geq 3$, $f_k = f_{k-1} + f_{k-2}$.

(a) Prove by induction on n that $f_n < \left(\frac{7}{4}\right)^{n-1}$ for all integers $n \geq 2$.

(b) Prove by induction on n that $\sum_{i=1}^{n} {f_i}^2 = f_{n+1}f_n$ for all integers $n \geq 1$.

(c) Prove by induction on n that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all integers $n \geq 1$.