MATHEMATICS 271 L20 SPRINGER 2005
MIDTERM EXAMINATION SOLUTION

1. Write the negations of the following statements:
(a) For all integers a, b and c, if $a \mid (b+c)$ then $a \mid b$ and $a \mid c$.
(b) There exists an integer x so that for all integers y, if $x \mid y^2$ then $x \mid y$.
(c) For all real numbers x, if x is rational then there exists an integer m so that mx is an integer.
Solution:
(a) There exist integers a, b and c so that $a \mid (b+c)$, but $a \nmid b$ or $a \nmid c$.
(b) For all integers x, there exists an integer y so that $x \mid y^2$ but $x \nmid y$.

2. Which of the following are true? No explanation needed.
(a) There exist real numbers x and y so that $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Answer: **True**

(b) There exist real numbers x and y so that $\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$. Answer: **True**

(c) 0 is divisible by 0. Answer: **True**

(d) $(-2005) \bmod 3 = 1$. Answer: **False**

3. Let $\mathcal{P}$ be the statement: "For all integers a and b, if $a \mid b^2$ then $a \mid b$."
(a) Is $\mathcal{P}$ true? Prove your answer.
(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true? Prove your answer.
(c) Write the contrapositive of $\mathcal{P}$. Is the contrapositive of $\mathcal{P}$ true? Prove your answer.
Solution:
(a) $\mathcal{P}$ is false because when $a = 4$ and $b = 2$ we see that $a \mid b^2$ (because $4 \mid 4$), but $a \nmid b$ (because $4 \nmid 2$).
(b) The converse of $\mathcal{P}$ is: "For all integers a and b, if $a \mid b$ then $a \mid b^2$."
The converse of $\mathcal{P}$ is true and here is a proof. Let a, b be integers so that $a \mid b$. Since $a \mid b$, there is an integer k so that $b = ak$, and therefore, $b^2 = (ak)\, b = a\,(kb)$ which implies that $a \mid b^2$.
(c) The contrapositive of $\mathcal{P}$ is: "For all integers a and b, if $a \nmid b$ then $a \nmid b^2$."
The contrapositive of $\mathcal{P}$ is false because it is logically equivalent to $\mathcal{P}$ which is false.

4. Prove or disprove the following:
(a) For all sets A, B and C, $A - (B - C) \subseteq (A - B) - C$.
(b) For all sets A, B and C, $(A - B) - C \subseteq A - (B - C)$.
Solution:
(a) This statement is false. For example, when $A = C = \{1\}$ and $B = \emptyset$, we have $A - (B - C) = A - (\emptyset - C) = A - \emptyset = A = \{1\} \neq \emptyset = \{1\} - \{1\} = (\{1\} - \emptyset) - \{1\} = (A - B) - C$.
(b) This statement is true and here is a proof. Let A, B and C be sets. Let $x \in (A - B) - C$. Since $x \in (A - B) - C$, we have that $x \in (A - B)$ and $x \notin C$. Since $x \in (A - B) - C$, we

see that $x \in A$. Since $x \notin C$, we get that $x \notin (B - C)$. Since $x \in A$ and $x \notin (B - C)$, we have that $x \in A - (B - C)$. Thus, we have proved that $(A - B) - C \subseteq A - (B - C)$.

5. Prove by induction on n that $\sum_{i=1}^{n-1} i\,(i+1) = \frac{n\,(n-1)\,(n+1)}{3}$ for all integers $n \geq 2$.

Solution:

Basis ($n = 1$): $\sum_{i=1}^{2-1} i\,(i+1) = 1 \times (1+1) = 1 \times 2 = 2 = \frac{6}{3} = \frac{2 \times 1 \times 3}{3} = \frac{2\,(2-1)\,(2+1)}{3}$

Inductive Step: Let $k \geq 2$ be an integer, and suppose that

$$\sum_{i=1}^{k-1} i\,(i+1) = \frac{k\,(k-1)\,(k+1)}{3} \qquad [IH]$$

We want to show that $\sum_{i=1}^{k} i\,(i+1) = \frac{(k+1)\,k(k+2)}{3}$

Now,

$$\begin{aligned}
\sum_{i=1}^{k} i\,(i+1) &= \left[\sum_{i=1}^{k-1} i\,(i+1)\right] + k\,(k+1) \\
&= \frac{k\,(k-1)\,(k+1)}{3} + k\,(k+1) \qquad \text{by } [IH] \\
&= \frac{k\,(k-1)\,(k+1) + 3k\,(k+1)}{3} \\
&= \frac{(k\,(k-1) + 3k)\,(k+1)}{3} \\
&= \frac{(k^2 + 2k)\,(k+1)}{3} \\
&= \frac{k(k+2)\,(k+1)}{3} \\
&= \frac{(k+1)\,k(k+2)}{3}
\end{aligned}$$

Thus, by the principle of Mathematical Induction,

$\sum_{i=1}^{n-1} i\,(i+1) = \frac{n\,(n-1)\,(n+1)}{3}$ for all integers $n \geq 2$.