**Midterm Exam for Math 271**

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**Instructions.** You have 2 hours to complete this test. Please provide detailed solutions for the exercises. Only complete answers with sufficient explanation are worth full credit. No open textbooks or notes are allowed.

1. (6 marks) Deteremine whether the following statement from is a tautology or a contradiction.

$$(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $p \wedge \sim q$ | $\sim p \vee (p \wedge \sim q)$ | $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | T |
| T | F | F | T | F | T | T | T |
| F | T | T | F | F | F | T | T |
| F | F | T | T | F | F | T | T |

tautology

2. (6 marks) Prove the statement. For any integer $n$, $n^2 - 2$ is not divisible by 4.

Case 1: Suppose that $n$ is odd. Then $n = 2k+1$ for some integer $k$. $n^2 - 2 = (2k+1)^2 - 2 = 4k^2 + 4k + 1 - 2 =$
$= 4k^2 + 4k - 1 = 4(k^2 + k) - 1$. Clearly, $4(k^2+k) - 1$ is not divisible by 4.

Case 2: Suppose that $n$ is even. Then $n = 2k$ for some integer $k$. $n^2 - 2 = (2k)^2 - 2 = 4k^2 - 2 = 2(2k^2 - 1)$.
Notice that $2k^2 - 1$ is always odd, so $4k^2 - 2$ is not divisible by 4. $\square$

3. (6 marks) Negate the statements.

a) $\forall n \in \mathbf{Z}$, if $n$ is a prime then $n$ is odd or $n = 2$.

b) $\forall$ integers $d$, if $d/6$ is an integer then $d = 3$.

a) $\exists n \in \mathbb{Z}$ such that $n$ is a prime and $n$ is neither odd nor $n = 2$.

b) $\exists$ an integer $d$ such that $d/6$ is an integer and $d \neq 3$.

4. (4 marks) When an integer $a$ is divided by 7, the remainder is 4. What is the remainder when $5a$ is divided by 7?

$\exists\, b \in \mathbb{Z}$ such that $a = 7b + 4$

Then $5a = 5(7b+4) = 35b + 20 =$

$= 35b + 14 + 6 = 7(5b+2) + 6.$

Therefore the remainder is 6.

5. (8 marks) Prove the following statement by induction.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integers $n \geq 1$.

Step 1: $n = 1$ $\quad \frac{1}{1 \cdot 2} = \frac{1}{1+1} = \frac{1}{2}$ ✓

step 2 $k \geq 1$, assume that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

We must show that $\frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

$LHS = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} =$

$= \frac{(k+1)^{\cancel{2}}}{(\cancel{k+1})(k+2)} = \frac{k+1}{k+2} = RHS$ ✓

6. (8 marks) Prove by induction that $n^3 - 7n + 3$ is divisible by 3 for each integer $n \geq 0$.

Step 1: $n = 0$, $0^3 - 7 \cdot 0 + 3 = 3$ ✓

Step 2: $k \geq 0$, assume that $3 \mid k^3 - 7k + 3$.

We must show that $3 \mid (k+1)^3 - 7(k+1) + 3$.

$(k+1)^3 - 7(k+1) + 3 = k^3 + 3k^2 + 3k + 1 - 7(k+1) + 3 =$

$= (k^3 - 7k + 3) + 3k^2 + 3k - 6 = (k^3 - 7k + 3) + 3(k^2 + k - 2)$

$3 \mid$ by Induction hypothesis $\qquad 3(k^2 + k - 2) = 3b$, $3 \mid$

7. (6 marks) Show that for all sets $A, B$, and $C$,

$$(A - B) \cup (C - B) = (A \cup C) - B.$$

"$(A-B) \cup (C-B) \subseteq (A \cup C) - B$"

$x \in (A-B) \cup (C-B) \Rightarrow x \in A-B$ or $x \in C-B \Rightarrow x \notin B$ and $x \in A$ or $x \in C \Rightarrow$

$\Rightarrow x \notin B$ and $x \in A \cup C \Rightarrow x \in (A \cup C) - B$ ✓

"$(A-B) \cup (C-B) \supseteq (A \cup C) - B$"

$x \in (A \cup C) - B \Rightarrow x \in A \cup C$ and $x \notin B \Rightarrow x \in A$ or $x \in C$ and $x \notin B \Rightarrow$

$\Rightarrow x \in A-B$ or $x \in C-B \Rightarrow x \in (A-B) \cup (C-B)$ ✓

8. (6 marks) Prove the statement. For all sets $A, B$, and $C$, if $B \cap C \subseteq A$ then $(C - A) \cap (B - A) = \emptyset$.

On the contrary, assume that $x \in (C-A) \cap (B-A)$.

Then $x \in C-A$ and $x \in B-A \Rightarrow x \in C$ and $x \in B$ and $x \notin A$

$\Rightarrow x \in B \cap C$ and $x \notin A \Rightarrow x \in A$ and $x \notin A$ $\lightning$

contradiction.

Therefore, our assumption to the contrary is false and so the statement is true.

**Bonus Qeustion:** (3 marks) Prove the following statement by induction.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all integers } n \geq 1.$$

Step 1: $n=1$ $\quad 1^2 = \frac{1 \cdot (1+1)(2+1)}{6} = \frac{2 \cdot 3}{6} = 1$ ✓

Step 2: $k \geq 1$, assume that $1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

we must show that $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

LHS: $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$

$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right] = \frac{k+1}{6}\left[2k^2 + k + 6k + 6\right] =$

$= \frac{k+1}{6}\left[2k^2 + 7k + 6\right] = \frac{k+1}{6}\left[(k+2)(2k+3)\right] = \text{RHS}$ ✓