

Quiz 3 for Math 271

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Instructions. You have 35 minutes to complete this quiz. Please provide detailed solutions for the exercises. Only complete answers with sufficient explanation are worth full credit. No open textbooks or notes are allowed.

1. (7 marks) Prove the following statement by induction.

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all integers } n \geq 2.$$

Step 1: $n=2$ $\left(1 - \frac{1}{2^2}\right) = \frac{2+1}{2 \cdot 2} = \frac{3}{4}$ ✓

Step 2: $k \geq 2$ Assume that $\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$.

$$\begin{aligned} \left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{1}{(k+1)^2}\right) &= \frac{k+1}{2k} \cdot \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2} \\ &= \frac{k+1}{2k} \cdot \frac{(k+1-1)(k+1+1)}{(k+1)^2} = \frac{(k+1)(k+2)}{2k(k+1)} = \frac{k+2}{2k+1} \quad \checkmark \quad \square \end{aligned}$$

2. (7 marks) Prove the following statement by induction.

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}, \text{ for all integers } n \geq 2.$$

Step 1: $n=2$ $\sqrt{2} < 1 + \frac{1}{\sqrt{2}}$ ✓

Step 2: $k \geq 2$ Assume that $\sqrt{k} < 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}$.

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} >$$

$$> \frac{\sqrt{k^2 + 1} + 1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} \quad \checkmark$$

□

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3. (6 marks) Prove the statement. For all sets $A, B,$ and C

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

$$\begin{aligned} \text{"} \subseteq \text{"} \quad & (x, y) \in A \times (B \cup C) \Rightarrow x \in A \text{ and } y \in B \text{ or } C \\ & \Rightarrow (x, y) \in A \times B \text{ or } (x, y) \in A \times C. \end{aligned}$$

$$\begin{aligned} \text{"} \supseteq \text{"} \quad & (x, y) \in (A \times B) \cup (A \times C) \Rightarrow x \in A \text{ and } y \in B \text{ or } y \in C \Rightarrow \\ & \Rightarrow x \in A \text{ and } y \in B \cup C \Rightarrow (x, y) \in A \times (B \cup C) \end{aligned}$$

Marks:

1).....

2).....

3).....

total:.....

