THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATHEMATICS 271 L(60) - Summer, 2005

August 18, 2005 *Time: 3 hours*

I.D. NUMBER	SURNAME	OTHER NAMES
	FODOR	

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary student I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an acceptable alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.
A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. A student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.

EXAMINATION RULES

1. Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.
2. No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.
3. All enquiries and requests must be addressed to supervisors only.
4. Candidates are strictly cautioned against:
   (a) speaking to other candidates or communicating with them under any circumstances whatsoever;
   (b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;
   (c) making use of calculators and/or portable computing machines not authorized by the instructor;
   (d) leaving answer papers exposed to view;
   (e) attempting to read other students' examination papers.

   The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.
5. Candidates are requested to write on both sides of the page, unless the examiner has asked that the left half page be reserved for rough drafts or calculations.
6. Discarded matter is to be struck out and not removed by mutilation of the examination answer book.
7. Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.
8. The candidate is to write his/her name on each answer book as directed and is to number each book.
9. During the examination a candidate must report to a supervisor before leaving the examination room.
10. Candidates must stop writing when the signal is given. Answer books must be handed to the supervisor-in-charge promptly. Failure to comply with these regulations will be cause for rejection of an aswer paper.
11. If during the course of an examination a student becomes ill or receives word of domestic affliction, the student must report at once to the supervisor, hand in the unfinished paper and request that it be cancelled. If physical and/or emotional ill health is the cause, the student must report at once to a physician/counsellor so that subsequent application for a deferred examination is supported by a completed Physical/Counsellor Statement form. Students can consult professionals at University Health Services or Counselling and Student Development Centre during normal working hours or consult their physician/counsellor in the community. **Once an examination has been handed in for marking a student cannot request that the examination be cancelled for whatever reason. Such a request will be denied. Retroactive withdrawals will also not be considered.**

Question	Total Marks	Actual Marks
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

NOTE: A calculator/formula sheet *IS NOT* allowed.

1. Write the contrapositive and the converse for the following statements.

   (a) [4 marks] If x is nonnegative, then x is positive or x is 0.

Converse: If x is positive or $x=0$, then x is nonnegative.

contrapositive: If x is not positive and $x \neq 0$, then x is ~~positive~~ negative.

   (b) [6 marks] For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$

Converse: For all sets A, B, if $A \cap B^c = \emptyset$, then $A \subseteq B$.

contrapositive: For all sets A, B, if $A \cap B^c \neq \emptyset$ then $A \nsubseteq B$.

2. [10 marks] Prove or disprove. For all integers a, b, and c, if $a \mid (b+c)$ then $a \mid b$ or $a \mid c$.

The statement is false.

Counterexample: $a = 4$

$b = c = 10$

$4 \mid 10 + 10 = 20$ but $4 \nmid 10$

3. (a) [3 marks] Determine whether the following function is one-to-one and justify your answer.

$$f(x) = \frac{x}{x^2+1}, \quad \text{for all real numbers } x.$$

No, the function is not one-to-one.

$x_1 = 2 \quad f(x_1) = f(2) = \frac{2}{2^2+1} = \frac{2}{5}$

$x_2 = \frac{1}{2} \quad f(x_2) = f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2+1} = \frac{\frac{1}{2}}{\frac{1}{4}+1} = \frac{\frac{1}{2}}{\frac{5}{4}} = \frac{2}{5}$

$x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

(b) [3 marks] How many onto functions are there from a set with ~~three~~ five elements to a set with ~~five~~ three elements?

total # of functions $= 3^5$

there are $\binom{3}{2} = 3$ ways to select 2 elements from a set of three

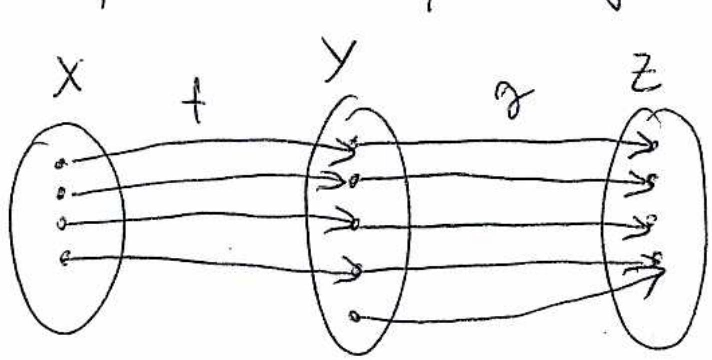
# of functions whose range consists of 2 elts $= \cancel{4}\ 2^5$.

But we counted the functions whose range is one element twice.

# of onto functions: $3^5 - \binom{3}{2}2^5 + 3 = 243 - 96 + 3 = \underline{150}$

(c) [4 marks] If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f$ is one-to-one, must both f and g be one-to-ne? Prove or give a counterexample.

No, look at the following counterexample.



g is clearly not one-to-one.

4. (a) [3 marks] Suppose that a graph has vertices of degree $0, 2, 2, 3$, and 9. How many edges does the graph have?

$\deg(G) = 2 \cdot \#$ of edges.

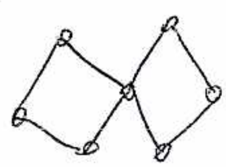
$\deg(G) = 0+2+2+3+9 = 16$

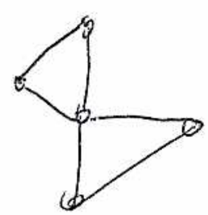
$\#$ of edges $= \underline{\underline{8}}$

(b) [3 marks] Draw a graph with four vertices and degrees $1, 2, 3$, and 3, or explain why such a graph does not exist.

There is no such graph, because the number of odd degree vertices must be even.

(c) [4 marks] Give two examples of graphs that have Euler circuits but not Hamiltonian circuits.





5. Suppose that in a certain state, all automobile licence plates have four letters followed by three digits.

(a) [5 marks] How many different licence plates are possible?

26 26 26 26 10 10 10

$\underline{\underline{26^4 \cdot 10^3}}$

(b) [5 marks] How many licence plates are possbile in which all the letters and numbers are different?

26 25 24 23 10 9 8

$\underline{\underline{26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8}}$

6. (a) [5 marks] How many integers from 1 through 9999 do not have any repeated digits?

\# of 1-digit numbers with no rep. digits between 1 and 9999 = 9

\# of 2-digit — " — $= 9 \cdot 9 = 8$

\# of 3-digit — " — $= 9 \cdot 9 \cdot 8 = 648$

\# of 4-digit — " — $= 9 \cdot 9 \cdot 8 \cdot 7 = 4536$

total = $\underline{\underline{5274}}$

(b) [5 marks] What is the probability that an integer chosen at random from 1 through 9999 has at least one repeated digit?

$$\frac{9999 - 5274}{9999} \approx 47\%$$

7. (a) [5 marks] How many 16-bit strings contain exactly seven 1's?

Once we choose the positions of 1's in the string the positions of 0's are determined. Thus, the # of 16-bit strings with exactly 7 1's is

$\underline{\underline{\binom{16}{7}}}$.

(b) [5 marks] How many 16-bit strings contain at least one 1?

There is only one 16-bit string with no 1 in it, the one with only 0's. Therefore the answer is as follows:

total # of 16-bit strings: 2^{16}

\# of 16-bit string with no 1 : $\underline{1}$

answer: $2^{16} - 1$

8. [10 marks] Determine whether the following relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

   D is the binary relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}$, $xDy \Leftrightarrow xy \geq 0$.

reflexivity: $x \in \mathbb{R} \Rightarrow x \cdot x = x^2 \geq 0 \Rightarrow xDx$. ✓

symmetry: $x, y \in \mathbb{R}$ and $xDy \Rightarrow x \cdot y \geq 0 \Rightarrow y \cdot x \geq 0 \Rightarrow$
$\Rightarrow yDx$. ✓

transitivity: No, D is not transitive.

$x = -1 \quad -1 \cdot 0 = 0 \Rightarrow xDy$
$y = 0 \quad 0 \cdot 1 = 0 \Rightarrow yDz$
$z = 1 \quad -1 \cdot 1 = -1 < 0 \Rightarrow x \not{D} z$

9. [10 marks] Prove that the following relation is an equivalence relation.

   I is the relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}$, $xIy \Leftrightarrow x - y$ is an integer.

reflexivity: $x \in \mathbb{R} \Rightarrow x - x = 0$ integer $\Rightarrow xIx$ ✓

symmetry: $x, y \in \mathbb{R}$ and $xIy \Rightarrow x - y = n$ integer $\Rightarrow$
$\Rightarrow y - x = -n$ integer $\Rightarrow yIx$. ✓

transitivity: $x, y, z \in \mathbb{R}$, xIy, $yIz \Rightarrow x - y = n$ integer and $z - y = m$ integer $\Rightarrow x - z = x - y + y - z$
$= n - m$ integer $\Rightarrow xIz$.

10. [10 marks] Prove by induction that $1 + nx \leq (1+x)^n$, for all real numbers $x > -1$ and integers $n \geq 2$.

If $x = 0$ then $1 + 0 \leq (1+0)^n$ holds for all $n \geq 2$.

Let $x \neq 0$ and $x > -1$.

Base Case: $n = 2$

$$1 + 2x \leq 1 + 2x + x^2 = (1+x)^2$$

General Case: $k \geq 2$ and assume that $(1+kx) \leq (1+x)^k$.

We must show that $1 + (k+1)x \leq (1+x)^{k+1}$.

$$RHS = (1+x)^{(k+1)} = (1+x)^k(1+x) = (1+x)^k + x(1+x)^k \geq$$

$$\geq (1+kx) + x(1+kx) = 1 + kx + x + kx^2 =$$

$$= 1 + (k+1)x + kx^2 \geq 1 + (k+1)x.$$

End of Examination