University of Calgary
Department of Mathematics and Statistics

MATH 271 (L60)

Date of exam **MIDTERM** **Duration of exam**
July 25, 2007 **2 hours**

STUDENT'S ID: Solution Key

INSTRUCTIONS: No calculators, open book or formula sheets.

1. [6 marks] Write the negation and the inverse of each statement.
a) If an integer is divisible by 2, then it is even.
b) $\forall$ animals x, if x is a dog then x has paws and x has a tail.
c) $\exists$ an integer x such that x^2 equals 4.

a) Negation: $\exists$ an integer n such that n is divisible by 2 but n is not even.
Inverse: If an integer is not divisible by 2, then it is not even.

b) Negation: $\exists$ an animal x such that x is a dog and x does not have paws or x does not have a tail.
Inverse: $\forall$ animals x, if x is not a dog then x does not have paws or x does not have a tail.

c) Negation: $\forall$ integers x, $x^2 \neq 4$.

2. [5 marks] Prove the following statement.

For all real numbers x, if x is not an integer, then $\lfloor x \rfloor + \lfloor -x \rfloor = -1$.

Let $x \in \mathbb{R}$ such that $x \notin \mathbb{Z}$. Then for some integer n,

$$n < x < n+1 \Rightarrow \lfloor x \rfloor = n \text{ by definition.}$$

$$-n > -x > -n-1 \Longrightarrow \lfloor -x \rfloor = -n-1$$

Thus, $\lfloor x \rfloor + \lfloor -x \rfloor = n + (-n-1) = -1$. $\square$

3. [5 marks] Determine whether the following statement is true or false. If it is true then prove it, if it is false then give a counterexample.

The square root of any irrational number is irrational.

TRUE

Proof: (by contradiction)

Assume the contrary: $\exists$ an irrational number r such that $\sqrt{r}$ is rational.

$\sqrt{r}$ rational $\Rightarrow \exists\, p, q \in \mathbb{Z},\ q \neq 0$ such that $\sqrt{r} = \frac{p}{q}$.

Then $r = (\sqrt{r})^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2},\ q^2 \neq 0,\ p^2, q^2 \in \mathbb{Z} \Rightarrow r$ rational

Therefore the assumption on the contrary is false and the statement is true. $\square$

4. [5 marks] Find an explicit formula for the following sum.

$5^3 + 5^4 + 5^5 + \cdots + 5^k$, where k is any positive integer with $k > 3$.

$$5^3 + 5^4 + \dots + 5^k = 5^3(5^0 + 5^1 + \dots + 5^{k-3}) =$$

$$= 5^3\left(\frac{5^{k-2} - 1}{5 - 1}\right) = \frac{125}{4} \cdot (5^{k-2} - 1)$$

5. [5 marks] Prove the following statement by mathematical induction.

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

Basis Case: $n=0$: $\sum_{i=1}^{0+1} i 2^i = 2 = 0 \cdot 2^{0+2} + 2$ ✓

Ind. Step: Assume that for some $n=k$, $k \geq 0$, $\sum_{i=1}^{k+1} i 2^i = k 2^{k+2} + 2$.

We need: $\sum_{i=1}^{k+2} i 2^i = (k+1) 2^{k+3} + 2$.

$$\sum_{i=1}^{k+2} i 2^i = \left(\sum_{i=1}^{k+1} i 2^i\right) + (k+2)2^{k+2} = \left(k 2^{k+2} + 2\right) + (k+2) 2^{k+2} =$$

$$= 2k 2^{k+2} + 2k^{k+2} + 2 = k 2^{k+3} + 2^{k+3} + 2 = (k+1) 2^{k+3} + 1$$

□

6. [6 marks] Prove the statement by mathematical induction.

For any integer $n \geq 0$, $7^n - 2^n$ is divisible by 5.

Basis Case: $n=0$: $7^0 - 2^0 = 1 - 1 = 0$ and $5 | 0$ ✓

Inductive Step: Assume that for some $n=k$, $k \geq 0$, $5 | 7^k - 2^k$.

We need: $5 | 7^{k+1} - 2^{k+1}$.

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = \underbrace{5 \cdot 7^k}_{5|} + 2\underbrace{(7^k - 2^k)}_{5|}$$

therefore $5 | 7^{k+1} - 2^{k+1}$. □

7. [6 marks] Prove the statement by mathematical induction.

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}, \text{ for all integers } n \geq 2.$$

Basis Case: $n=2$: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} > 2 > \sqrt{2}$. ✓

Ind. Step: Assume that for some $n=k$, $k \geq 2$, $\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}$.

We need: $\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k+1}}$.

$$RHS = \underbrace{\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}}_{> \sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k}} >$$

$$> \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$$ ✓ $\square$

8. [6 marks] Suppose that $g_0, g_1, \dots$ is a sequence defined as follows:
$g_0 = 12$, $g_1 = 29$,
$g_k = 5g_{k-1} - 6g_{k-2}$ for all integeres $k \geq 2$.
Prove that $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

Basis Case: $g_0 = 12 = 5 \cdot 3^0 + 7 \cdot 2^0 = 5+7$ ✓

$g_1 = 29 = 5 \cdot 3^1 + 7 \cdot 2^1 = 15 + 14$ ✓

Ind. Step: Assume that $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all $0 \leq n < k$ for some k.

We need: $g_k = 5 \cdot 3^k + 7 \cdot 2^k$.

$$g_k = 5g_{k-1} - 6g_{k-2} = 5(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) - 6(5 \cdot 3^{k-2} + 7 \cdot 2^{k-2}) =$$

$$= 25 \cdot 3^{k-1} + 35 \cdot 2^{k-1} - 30 \cdot 3^{k-2} - 42 \cdot 2^{k-2} = 45 \cdot 3^{k-2} + 28 \cdot 2^{k-2}$$

$$= 5 \cdot 3^k + 7 \cdot 2^k$$ $\square$

STUDENT'S NAME: SOLUTION KEY

9. [6 marks] Let the universal set be the set of all real numbers $\mathbb{R}$ and $A = \{x \in \mathbb{R} \mid 0 < x \leq 2\}$, $B = \{x \in \mathbb{R} \mid 1 \leq x < 4\}$ and $C = \{x \in \mathbb{R} \mid 3 \leq x < 9\}$. Find each of the following.

a) $A \cap C$
b) $A^c \cap B^c$
c) $(A \cap B)^c$

a) $A \cap C = \phi$

b) $A^c \cap B^c = (A \cup B)^c = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$

c) $(A \cap B)^c = \{x \in \mathbb{R} \mid x < 1 \text{ or } x > 2\}$.

MARKS:

1)........................
2)........................
3)........................
4)........................
5)........................
6)........................
7)........................
8)........................
9)........................
Total:....................