MATH 271 ASSIGNMENT 4 SOLUTIONS

1. For each positive integer n, let $[n] = \{1, 2, 3, \ldots, n\}$, and define

$$\mathcal{S}_\cup(n) = \text{ the set of all ordered pairs } (A, B) \text{ of sets such that } A \cup B = [n];$$
$$\mathcal{S}_\cap(n) = \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \cap B = \emptyset;$$
$$\mathcal{S}_\subseteq(n) = \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \subseteq B.$$

(a) Find $\mathcal{S}_\cup(1)$ and $\mathcal{S}_\cup(2)$.

(b) Prove that $\mathcal{S}_\cup(n)$ has exactly 3^n elements.

(c) Prove that $(A, B) \in \mathcal{S}_\cup(n)$ if and only if $(A^c, B^c) \in \mathcal{S}_\cap(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_\cap(n)$.

(d) Prove that $(A, B) \in \mathcal{S}_\cup(n)$ if and only if $(A^c, B) \in \mathcal{S}_\subseteq(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_\subseteq(n)$.

(a) We get

$$\mathcal{S}_\cup(1) = \big\{(\{1\}, \emptyset),\ (\emptyset, \{1\}),\ (\{1\}, \{1\})\big\}$$

and

$$\mathcal{S}_\cup(2) = \Big\{(\{1,2\}, \emptyset),\ (\emptyset, \{1,2\}),\ (\{1,2\}, \{1\}),\ (\{1\}, \{1,2\}),\ (\{1,2\}, \{2\}),$$
$$(\{2\}, \{1,2\}),\ (\{1,2\}, \{1,2\}),\ (\{1\}, \{2\}),\ (\{2\}, \{1\})\Big\}.$$

(b) We count how many ways there are to construct sets A and B so that $A \cup B = \{1, 2, \ldots, n\}$. To get this union, we need each number from 1 to n to either be in A, or in B, or in both. So we have three possibilities for each of the n numbers from 1 to n. Since these choices are all independent, there are $3 \cdot 3 \cdot \ldots \cdot 3 = 3^n$ such ordered pairs (A, B).

(c) First assume that $(A, B) \in \mathcal{S}_\cup(n)$. Then $A \cup B = [n]$, so by De Morgan's Law (page 272, #9(a)),

$$A^c \cap B^c = (A \cup B)^c = [n]^c = \emptyset,$$

therefore $(A^c, B^c) \in \mathcal{S}_\cap(n)$.

Conversely, assume that $(A^c, B^c) \in \mathcal{S}_\cap(n)$. Then $A^c \cap B^c = \emptyset$, so by various properties on page 272,

$$A \cup B = (A^c)^c \cup (B^c)^c = (A^c \cap B^c)^c = \emptyset^c = [n],$$

therefore $(A, B) \in \mathcal{S}_\cup(n)$.

This means that there is a one-to-one correspondence between the elements of $\mathcal{S}_\cup(n)$ and the elements of $\mathcal{S}_\cap(n)$, so by part (b) $\mathcal{S}_\cap(n)$ must also have 3^n elements.

(d) First assume that $(A, B) \in \mathcal{S}_\cup(n)$, which means $A \cup B = [n]$. We want to prove that $(A^c, B) \in \mathcal{S}_\subseteq(n)$, which means we want to prove that $A^c \subseteq B$. Let $x \in A^c$ be arbitrary. This means that $x \in [n]$ but $x \notin A$. Since $A \cup B = [n]$, $x \in [n]$ means $x \in A \cup B$, and since $x \notin A$ we conclude that $x \in B$. Therefore $A^c \subseteq B$ and $(A^c, B) \in \mathcal{S}_\subseteq(n)$.

Conversely, assume that $(A^c, B) \in \mathcal{S}_\subseteq(n)$, which means $A^c \subseteq B$. We want to prove that $(A, B) \in \mathcal{S}_\cup(n)$, which means we want to prove that $A \cup B = [n]$. Since $A \cup B \subseteq [n]$, we only need to prove that $[n] \subseteq A \cup B$. Let $x \in [n]$ be arbitrary. If $x \in A$, then $x \in A \cup B$ which is what we want. On the other hand, if $x \notin A$, then $x \in A^c$, and since $A^c \subseteq B$, this means that $x \in B$ and thus $x \in A \cup B$. So in either case we get that $x \in A \cup B$. Therefore $[n] \subseteq A \cup B$, so $A \cup B = [n]$, so $(A, B) \in \mathcal{S}_\cup(n)$.

Once again this means that there is a one-to-one correspondence between the elements of $\mathcal{S}_\cup(n)$ and the elements of $\mathcal{S}_\subseteq(n)$, so by part (b) $\mathcal{S}_\subseteq(n)$ must also have 3^n elements.

2. For each positive integer n, let $f(n)$ be the number of ordered pairs (A, B) of subsets of $\{1, 2, 3, \dots, n\}$ so that $A \cup B$ has an even number of elements.

   (a) Find $f(1)$ and $f(2)$ by listing all the ordered pairs of subsets.

   (b) Use Problem 1(b) to prove that for any n,

   $$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k}.$$

   Show that your answers to part (a) agree with this formula.

   (c) Mimic Example 6.7.4 on page 368 to prove that $\sum_{i=0}^{n} \binom{n}{i} 3^i = 4^n$ and thus

   $$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n - f(n).$$

   (d) Use Pascal's Formula (page 360), (b) and (c), and mathematical induction to prove that

   $$f(n) = \begin{cases} 2^{n-1}(2^n - 1) & \text{if } n \text{ is odd,} \\ 2^{n-1}(2^n + 1) & \text{if } n \text{ is even.} \end{cases}$$

(a) Since A and B are subsets of $\{1, 2, \dots, n\}$, we always have $A \cup B \subseteq \{1, 2, \dots, n\}$. So when $n = 1$, the only way for $A \cup B$ to have an even number of elements is if $A \cup B = \emptyset$, so the only ordered pair (A, B) that works is $(\emptyset, \emptyset)$, and thus $f(1) = \mathbf{1}$. When $n = 2$, we could have $A \cup B = \emptyset$ or $A \cup B = \{1, 2\}$, so the ordered pairs (A, B) that work are $(\emptyset, \emptyset)$ plus the nine ordered pairs in $\mathcal{S}_\cup(2)$ from problem 1(a). Thus $f(2) = \mathbf{10}$.

(b) First, from problem 1(b) it is clear that for any set S with m elements there must be exactly 3^m ordered pairs (A, B) of sets so that $A \cup B = S$ (since the names of the m elements of S don't matter). Let k be an integer so that $0 \le 2k \le n$. There are $\binom{n}{2k}$ subsets of $\{1, 2, \dots, n\}$ with $2k$ elements, and for each of these subsets there are 3^{2k} ordered pairs (A, B) of sets whose union is that subset. Thus for each k, there are $\binom{n}{2k} 3^{2k}$ ordered pairs (A, B) of subsets of $\{1, 2, \dots, n\}$ so that $A \cup B$ has $2k$ elements. Adding over all possible values of k (namely $k = 0, 1, \dots, \lfloor n/2 \rfloor$), we get that

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k}.$$

When $n = 1$ this says

$$f(1) = \sum_{k=0}^{0} \binom{1}{2k} 3^{2k} = \binom{1}{0} 3^0 = 1,$$

and when $n = 2$ it says

$$f(2) = \sum_{k=0}^{1} \binom{2}{2k} 3^{2k} = \binom{2}{0} 3^0 + \binom{2}{2} 3^2 = 1 + 9 = 10,$$

both agreeing with part (a).

(c) We put $a = 1$ and $b = 3$ into the Binomial Theorem (Theorem 6.7.1 on page 364) to get

$$\sum_{i=0}^{n} \binom{n}{i} 3^i = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 3^i = (1+3)^n = 4^n.$$

Splitting this sum into two parts, one with all the even i's and one with all the odd i's, we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n.$$

But the first sum is just $f(n)$ by part (b), so subtracting it from both sides gives us

$$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n - f(n)$$

as required.

(d) *Basis step.* When $n = 1$ (which is odd) the formula says $f(1) = 2^0(2^1 - 1) = 1$, which is correct by part (a).

Inductive step. Assume that the formula is correct for some integer $n \geq 1$. We want to prove it is correct for the next integer $n + 1$. Well,

$$\begin{aligned}
f(n+1) &= \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} 3^{2k} && \text{by part (b)} \\
&= \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \left[\binom{n}{2k} + \binom{n}{2k-1} \right] 3^{2k} && \text{by Pascal's Formula} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3 \cdot 3^{2k-1} \\
&= f(n) + 3(4^n - f(n)) && \text{by parts (b) and (c)} \\
&= 3(4^n) - 2f(n) \\
&= \begin{cases} 3(4^n) - 2^n(2^n - 1) & \text{if } n \text{ is odd} \\ 3(4^n) - 2^n(2^n + 1) & \text{if } n \text{ is even} \end{cases} && \text{by assumption} \\
&= \begin{cases} 2(4^n) + 2^n = 2^n(2^{n+1} + 1) & \text{if } n+1 \text{ is even} \\ 2(4^n) - 2^n = 2^n(2^{n+1} - 1) & \text{if } n+1 \text{ is odd,} \end{cases}
\end{aligned}$$

which completes the inductive step. Therefore the formula is correct for all integers $n \geq 1$.

Note: If n is odd, and if $2^n - 1$ happens to be a prime number, then the value $f(n) = 2^{n-1}(2^n - 1)$ is what is called a *perfect number*. To find out what these are, ask your professor or TA, or search the internet.

3. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer n.

   (a) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (b) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (c) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.

   (d) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.

   (a) Since, for every k, $f(k)$ must be one of the k values $1, 2, \ldots, k$, there is one choice for $f(1)$ (namely 1), two choices for $f(2)$ (namely 1 or 2), and so on up to n choices for $f(n)$ (namely any of $1, 2, \ldots, n$). Thus by the Multiplication Rule there are $1 \cdot 2 \cdot \ldots \cdot n = n!$ ways to assign all the values $f(1), f(2), \ldots, f(n)$, that is, $\mathbf{n!}$ different functions.

   (b) If f must be one-to-one, then we still must assign $f(1) = 1$, but then we cannot assign $f(2)$ to be 1 too, so we must put $f(2) = 2$. Next we cannot let $f(3)$ be 1 or 2, so we must put $f(3) = 3$. Continuing in this way, we are forced to put $f(k) = k$ for each k, so there is just **one** one-to-one function $f : [n] \to [n]$, namely the identity function.

   (c) Proceeding as in part (a), for each k, $f(k)$ must be one of the $k+1$ choices $1, 2, \ldots, k+1$, provided that $k < n$. So $f(1)$ can be 1 or 2, $f(2)$ can be 1, 2 or 3, and so on up to $f(n-1)$ which can be any of $1, 2, \ldots, n$. But $f(n)$ must still belong to $[n]$ so there are only n choices for $f(n)$. Thus by the Multiplication Rule the total number of functions is $2 \cdot 3 \cdot \ldots \cdot n \cdot n = \mathbf{n(n!)}$.

   (d) Note that since $[n]$ is finite, a function $f : [n] \to [n]$ is onto if and only if it is one-to-one. So we are really just counting one-to-one functions again. Now $f(1)$ must be 1 or 2, so there are two choices for $f(1)$. Then $f(2)$ must be 1, 2 or 3, so removing whichever choice we made for $f(1)$ will leave two choices for $f(2)$. In general there will be $k + 1$ choices for $f(k)$ (namely $1, 2, \ldots, k + 1$), but after we remove the choices we make for $f(1), f(2), \ldots, f(k-1)$ we will always have exactly two choices left for $f(k + 1)$. The exception again is that for $f(n)$ there are only n choices originally (namely $1, 2, \ldots, n$), and after we remove the choices we make for $f(1), f(2), \ldots, f(n-1)$ we will only have one choice left for $f(n)$. So in total there will be $2 \cdot 2 \cdot \ldots \cdot 2 \cdot 1 = \mathbf{2^{n-1}}$ onto functions.