MATH 271 ASSIGNMENT 5

Due 4:00 PM Friday, April 8, 2005. Put your assignment in the appropriate **wooden slot** (corresponding to your lecture section and last name) inside room MS 315. Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. If $f : \mathbf{R} \to \mathbf{R}$ is a function (where $\mathbf{R}$ is the set of all real numbers), we define the function $f^{(2)}$ to be the composition $f \circ f$, and for any integer $n \geq 2$, define $f^{(n+1)} = f \circ f^{(n)}$. So $f^{(2)}(x) = (f \circ f)(x) = f(f(x))$, $f^{(3)}(x) = (f \circ f^{(2)})(x) = f(f(f(x)))$, and so on.

   Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x^2$ for all $x \in \mathbf{R}$.

   (a) Find and simplify $f^{(2)}(x)$ and $f^{(3)}(x)$.

   (b) Use part (a) (and more calculations if you need them) to guess a formula for $f^{(n)}(x)$.

   (c) Prove your guess using mathematical induction.

   (d) Find all $x \in \mathbf{R}$ so that $f^{(271)}(x) = x$.

2. Let $[n] = \{1, 2, \ldots, n\}$, where n is a positive integer. Let $\mathcal{R}$ be the relation on the power set $\mathcal{P}([n])$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{R}B$ if and only if $1 \notin A - B$.

   (a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Explain.

   (b) Find the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ such that $A\mathcal{R}B$. [*Hint*: first count the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ so that $A \not\mathcal{R} B$.]

   (c) Suppose you choose sets $A, B \in \mathcal{P}([n])$ at random. What is the probability that $A\mathcal{R}B$?

   (d) Let $\mathcal{S}$ be the relation on the power set $\mathcal{P}(X)$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{S}B$ if and only if $1 \in A - B$. Is $\mathcal{S}$ transitive? Explain.

3. Let $\mathcal{F}$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, where n is a positive integer. Define a relation R on $\mathcal{F}$ by: for $f, g \in \mathcal{F}$, fRg if and only if $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$.

   (a) Prove that R is an equivalence relation on $\mathcal{F}$.

   (b) Suppose that $n = 2m+1$ is odd. Find the number of functions in the equivalence class $[id]$, where id is the identity function on $\{1, 2, \ldots, n\}$. How many of these functions are one-to-one and onto?

   (c) Suppose that $n = 2m$ is even. Find the number of functions in the equivalence class $[g]$, where $g(x) = 1$ is a constant function. How many of these functions are one-to-one and onto?