MATH 271 ASSIGNMENT 1 SOLUTIONS

1.

(a) Let $\mathcal{S}$ be the statement

For all integers n, if n is even then $3n - 11$ is odd.

Is $\mathcal{S}$ true? Give a proof or counterexample.

(b) Write out the *contrapositive* of statement $\mathcal{S}$, and give a proof or disproof.

(c) Write out the *converse* of statement $\mathcal{S}$, and give a proof or disproof.

(d) Prove or disprove the statement

For all integers n, if n is odd then $2n - 11$ is even.

Then write out the converse of this statement and prove or disprove it.

(a) $\mathcal{S}$ is **true**. Here is a proof.

Let n be an arbitrary even integer. This means that $n = 2k$ for some integer k. Then

$$3n - 11 = 3(2k) - 11 = 6k - 11 = 2(3k - 6) + 1$$

where $3k - 6$ is an integer. Therefore $3n - 11$ is odd by the definition of odd.

(b) The contrapositive of $\mathcal{S}$ is:

For all integers n, if $3n - 11$ is not odd then n is not even,

which could also be written (using a result on page 159 of the text)

For all integers n, if $3n - 11$ is even then n is odd.

It is **true**, because it is equivalent to the original statement $\mathcal{S}$ which is true.

(c) The converse of $\mathcal{S}$ is

For all integers n, if $3n - 11$ is odd then n is even.

This statement is **true**. Here is a proof.

Assume that $3n - 11$ is odd, where n is an integer. This means that $3n - 11 = 2k + 1$ for some integer k. We can rewrite this equation as $n = 2k + 12 - 2n = 2(k + 6 - n)$, where $k + 6 - n$ is an integer since k and n are integers. Therefore n equals 2 times an integer, so n is even.

Note. The converse could also be proven by writing its contrapositive

For all integers n, if n is not even then $3n - 11$ is not odd

in the form

For all integers n, if n is odd then $3n - 11$ is even

and proving this.

(d) This statement is **false**. A counterexample is $n = 1$. For then n is odd, but $2n - 11 = 2 - 11 = -9$ is not even.

The converse of this statement is

For all integers n, if $2n - 11$ is even then n is odd.

This statement is **true** vacuously. For every integer n, $2n - 11 = 2(n - 6) + 1$ where $n - 6$ is an integer, thus $2n - 11$ is odd and so cannot be even. Since the "if" part of the conditional never holds, the statement is true vacuously.

2. Prove or disprove the following statements:

(a) There exists a prime number a such that $a + 271$ is prime.

(b) There exists a prime number a such that $a + 271$ is composite.

(c) There exists a composite number a such that $a + 271$ is prime.

(d) There exists a composite number a such that $a + 271$ is composite.

(e) Choose one of statements (a) to (d) (your choice), replace 271 with your U of C ID number, and prove or disprove the resulting statement.

(a) This statement is **false**. Here is a proof.

Assume a is a prime number. We have two cases.

Case (i): $a = 2$. Then $a + 271 = 2 + 271 = 273 = 3 \cdot 91$, so $a + 271$ is not prime.

Case (ii): $a > 2$. Then a must be odd, so $a = 2k + 1$ for some integer k. Then $a + 271 = 2k + 1 + 271 = 2k + 272 = 2(k + 136)$, where $k + 136$ is an integer. Therefore $a + 271$ is not prime.

In neither case can we get that $a + 271$ is prime, so the statement is false.

(b) This statement is **true**. An example is $a = 3$. Then a is prime and $a + 271 = 274 = 2 \cdot 137$ is composite.

(c) This statement is **true**. An example is $a = 6$. Then a is composite and $a + 271 = 277$ is prime (it turns out).

Note. An alternate proof would go like this: since there are infinitely many primes (Theorem 3.7.4 of the text), there must be a prime $p \geq 275$. Then p is odd, so $p - 271$ must be even (prove it), and $p - 271 \geq 4$, so $p - 271$ is composite. Put $a = p - 271$; then a is composite and $a + 271 = p$ is prime.

(d) This statement is **true**. An example is $a = 9$. Then a is composite and $a + 271 = 280$ is composite too.

(e) Regardless of what your ID number is, probably (d) is the easiest statement to prove. Let's do it for the hypothetical ID number 123456. Choosing $a = 4$, we get that a is composite and that $a + 123456 = 123460$ is also composite.

3. *Note*: $\mathbf{Z}$ denotes the set of all integers, and $\mathbf{Z}^+$ denotes the set of all positive integers.

(a) Prove the following statements:

(i) $\exists a \in \mathbf{Z}$ so that $\forall b \in \mathbf{Z}, (a-b)|(a+b)$.

(ii) $\forall a \in \mathbf{Z}^+ \, \exists b \in \mathbf{Z}^+$ so that $(a-b)|(a+b)$.

(iii) $\forall a \in \mathbf{Z}^+ \, \exists b \in \mathbf{Z}^+$ so that $(a+b)|(a-b)$.

(b) Write out the *negation* of the following statement:

$$\forall a, b \in \mathbf{Z}^+, \text{ if } a|2 \text{ and } b|3 \text{ then } (a+b)|5.$$

Then show that the negation is true, so that the original statement is false.

(c) Prove the following statement:

$$\exists N \in \mathbf{Z}^+ \text{ so that } \forall a, b \in \mathbf{Z}^+, \text{ if } a|2 \text{ and } b|3 \text{ then } (a+b)|N.$$

(a) (i) Choose $a = 0$. Then the statement to be proved is: $\forall b \in \mathbf{Z}, (-b)|b$. To prove this, let b be an arbitrary integer. Then $b = (-b)(-1)$ where -1 is an integer, so $(-b)|b$.

(ii) Let a be an arbitrary positive integer. We need to find a positive integer b (maybe depending on a) so that $(a-b)|(a+b)$. Choose $b = a+1$, which is a positive integer. Then $a - b = -1$ and $a + b = 2a + 1$, so we need to show that $(-1)|(2a+1)$. But this is clear, since $2a + 1 = (-1)(-2a-1)$ where $-2a-1$ is an integer.

(iii) Let a be an arbitrary positive integer. We need to find a positive integer b (maybe depending on a) so that $(a+b)|(a-b)$. Choose $b = a$, which is a positive integer. Then $a + b = 2a$ and $a - b = 0$, so we need to show that $(2a)|0$. But this is clear, since $0 = 0 \cdot 2a$.

(b) The negation is:

$$\exists a, b \in \mathbf{Z}^+ \text{ so that } a|2 \text{ and } b|3 \text{ but } (a+b) \nmid 5.$$

This statement is true. For example we can choose $a = 1$ and $b = 1$; then $a|2$ and $b|3$ are both true, but $a + b = 2$, and $2 \nmid 5$.

(c) For $a|2$ we need either $a = 1$ or $a = 2$, and for $b|3$ we need either $b = 1$ or $b = 3$. Thus we will need N to satisfy all of the following:

- $(1+1)|N$, which says $2|N$;
- $(2+1)|N$, which says $3|N$;
- $(1+3)|N$, which says $4|N$;
- $(2+3)|N$, which says $5|N$.

So for example, $N = 2 \cdot 3 \cdot 4 \cdot 5 = \mathbf{120}$ will work. Actually $N = 3 \cdot 4 \cdot 5 = \mathbf{60}$ will work too, and this is the smallest value of N which will work.