MATH 271 ASSIGNMENT 5 SOLUTIONS

1. If $f : \mathbf{R} \to \mathbf{R}$ is a function (where $\mathbf{R}$ is the set of all real numbers), we define the function $f^{(2)}$ to be the composition $f \circ f$, and for any integer $n \geq 2$, define $f^{(n+1)} = f \circ f^{(n)}$. So $f^{(2)}(x) = (f \circ f)(x) = f(f(x))$, $f^{(3)}(x) = (f \circ f^{(2)})(x) = f(f(f(x)))$, and so on.

   Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x^2$ for all $x \in \mathbf{R}$.

   (a) Find and simplify $f^{(2)}(x)$ and $f^{(3)}(x)$.

   (b) Use part (a) (and more calculations if you need them) to guess a formula for $f^{(n)}(x)$.

   (c) Prove your guess using mathematical induction.

   (d) Find all $x \in \mathbf{R}$ so that $f^{(271)}(x) = x$.

   (a) We get
   $$f^{(2)}(x) = f(f(x)) = f(3x^2) = 3(3x^2)^2 = 3^3x^4$$
   and
   $$f^{(3)}(x) = f(f^{(2)}(x)) = f(3^3x^4) = 3(3^3x^4)^2 = 3^7x^8.$$

   (b) Since $f^{(2)}(x) = 3^3x^4 = 3^{2^2-1}x^{2^2}$ and $f^{(3)}(x) = 3^7x^8 = 3^{2^3-1}x^{2^3}$, we guess that
   $$f^{(n)}(x) = 3^{2^n-1}x^{2^n} \qquad \text{for all } n \geq 2.$$

   (c) *Basis step.* This is already done, since our formula is true when $n = 2$.
   Inductive step. Assume that $f^{(k)}(x) = 3^{2^k-1}x^{2^k}$ for some integer $k \geq 2$. We want to prove that $f^{(k+1)}(x) = 3^{2^{k+1}-1}x^{2^{k+1}}$. Well,
   $$\begin{aligned} f^{(k+1)}(x) &= f\left(f^{(k)}(x)\right) && \text{by definition} \\ &= f\left(3^{2^k-1}x^{2^k}\right) && \text{by assumption} \\ &= 3\left(3^{2^k-1}x^{2^k}\right)^2 \\ &= 3^{1+(2^k-1)2}x^{(2^k)2} \\ &= 3^{2^{k+1}-1}x^{2^{k+1}}, \end{aligned}$$
   which finishes the inductive step. This proves that our guess is correct for every integer $n \geq 2$.

   (d) Since $f^{(271)}(x) = 3^{2^{271}-1}x^{2^{271}}$, we need to solve the equation $3^{2^{271}-1}x^{2^{271}} = x$. One obvious solution is $x = \mathbf{0}$. So assuming now that $x \neq 0$, we can divide both sides by x and get $3^{2^{271}-1}x^{2^{271}-1} = 1$, which can be rewritten as $(3x)^{2^{271}-1} = 1$, which means that $3x = 1$ since $2^{271} - 1$ is odd. Thus the only other solution is $x = \mathbf{1/3}$.

2. Let $[n] = \{1, 2, \ldots, n\}$, where n is a positive integer. Let $\mathcal{R}$ be the relation on the power set $\mathcal{P}([n])$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{R}B$ if and only if $1 \notin A - B$.

   (a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Explain.

(b) Find the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ such that $A\mathcal{R}B$. [*Hint*: first count the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ so that $A \not\mathcal{R} B$.]

(c) Suppose you choose sets $A, B \in \mathcal{P}([n])$ at random. What is the probability that $A\mathcal{R}B$?

(d) Let $\mathcal{S}$ be the relation on the power set $\mathcal{P}([n])$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{S}B$ if and only if $1 \in A - B$. Is $\mathcal{S}$ transitive? Explain.

(a) $\mathcal{R}$ **is reflexive**. Here is a proof. Let $A \in \mathcal{P}([n])$ be arbitrary. Then $A - A = \emptyset$, so $1 \notin A - A$, so $A\mathcal{R}A$.

$\mathcal{R}$ **is not symmetric**. Here is a counterexample. Let $A = \emptyset$ and $B = \{1\}$. Then $A - B = \emptyset$, so $1 \notin A - B$, so $A\mathcal{R}B$. However $B - A = \{1\}$, so $1 \in B - A$, so $B \not\mathcal{R} A$.

$\mathcal{R}$ **is transitive**. Here is a proof. Let $A, B, C \in \mathcal{P}([n])$ be arbitrary so that $A\mathcal{R}B$ and $B\mathcal{R}C$. This means that $1 \notin A - B$ and $1 \notin B - C$. We want to prove that $A\mathcal{R}C$, which means we want to prove that $1 \notin A - C$. We do this by contradiction. Suppose that $1 \in A - C$. This means that $1 \in A$ but $1 \notin C$. Since $1 \in A$ but $1 \notin A - B$, it must mean that $1 \in B$. But now $1 \in B$ and $1 \notin C$ means $1 \in B - C$, which is a contradiction. Therefore $1 \notin A - C$, so $\mathcal{R}$ is transitive.

(b) Since $A \not\mathcal{R} B$ means $1 \in A - B$, to count the number of ordered pairs (A, B) so that $A \not\mathcal{R} B$ we just count the number of (A, B) so that $1 \in A$ and $1 \notin B$. The number of subsets A of $[n]$ so that $1 \in A$ is just the number of subsets of $\{2, 3, \ldots, n\}$, which is 2^{n-1} (for example, see p. 285). The number of subsets B of $[n]$ so that $1 \notin B$ is also just the number of subsets of $\{2, 3, \ldots, n\}$, which is 2^{n-1}. Thus the number of ordered pairs (A, B) so that $A \not\mathcal{R} B$ is $2^{n-1} \cdot 2^{n-1} = 2^{2(n-1)}$ by the Multiplication Rule. There are 2^n subsets of $[n]$ altogether, so there are $2^n \cdot 2^n = 2^{2n}$ ordered pairs (A, B) altogether. Therefore the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ such that $A\mathcal{R}B$ is $2^{2n} - 2^{2(n-1)} = 2^{2n-2}(2^2 - 1) = 3(2^{2n-2})$.

(c) Since all choices of subsets $A, B \in \mathcal{P}([n])$ are equally likely, the probability is

$$\frac{\text{number of } (A, B) \text{ so that } A\mathcal{R}B}{\text{total number of } (A, B)} = \frac{3(2^{2n-2})}{2^{2n}} = \frac{3}{4},$$

regardless of the value of n.

(d) Yes, $\mathcal{S}$ is transitive, **vacuously**. Suppose that $A, B, C \in \mathcal{P}([n])$ satisfy $A\mathcal{S}B$ and $B\mathcal{S}C$. This means that $1 \in A - B$ and $1 \in B - C$. But $1 \in A - B$ means in particular that $1 \notin B$, while $1 \in B - C$ means in particular that $1 \in B$. This is a contradiction, so the "if" part of the definition of transitivity can never happen, so the relation $\mathcal{S}$ is transitive vacuously.

3. Let $\mathcal{F}$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, where n is a positive integer. Define a relation R on $\mathcal{F}$ by: for $f, g \in \mathcal{F}$, fRg if and only if $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$.

   (a) Prove that R is an equivalence relation on $\mathcal{F}$.

   (b) Suppose that $n = 2m + 1$ is odd. Find the number of functions in the equivalence class $[id]$, where id is the identity function on $\{1, 2, \ldots, n\}$. How many of these functions are one-to-one and onto?

(c) Suppose that $n = 2m$ is even. Find the number of functions in the equivalence class $[g]$, where $g(x) = 1$ is a constant function. How many of these functions are one-to-one and onto?

(a) *R is reflexive.* Let $f \in \mathcal{F}$ be arbitrary. Then $f(k) + f(k) = 2f(k)$ is even for every $k \in \{1, 2, \ldots, n\}$, since $f(k)$ is an integer, so fRf.

R is symmetric. Let $f, g \in \mathcal{F}$ be arbitrary so that fRg. This means that $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$. But then $g(k) + f(k) = f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, so gRf.

R is transitive. Let $f, g, h \in \mathcal{F}$ be arbitrary so that fRg and gRh. This means that $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, and $g(k) + h(k)$ is even for all $k \in \{1, 2, \ldots, n\}$. But then $f(k) + g(k) + g(k) + h(k) = f(k) + h(k) + 2g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, so $f(k) + h(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, since the sum and difference of even numbers is even. Therefore fRh.

(b) We want to count the number of functions $f \in \mathcal{F}$ so that $fR\, id$. id is the function $id(k) = k$ for all $k \in \{1, 2, \ldots, n\}$. Thus $fR\, id$ means that $f(k) + k$ is even for all k. This in turn means that $f(k)$ must be even whenever k is even, and odd whenever k is odd. Since $n = 2m + 1$, there are m even numbers and $m + 1$ odd numbers in $\{1, 2, \ldots, n\}$. So for each of the m even k's there are m choices for $f(k)$, and for each of the $m + 1$ odd k's there are $m + 1$ choices for $f(k)$. Thus the total number of ways we can define f is $\mathbf{m^m(m+1)^{m+1}}$.

If we insist that f be one-to-one and onto, we only need to make it one-to-one, since any one-to-one function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ will have to be onto as well. Now when we count how many ways there are to define $f(2), f(4), \ldots, f(2m)$ (that is, $f(k)$ for the m even k's), we get m ways to define $f(2)$, $m - 1$ ways to define $f(4)$, and so on down to just one way to define $f(2m)$. So there are $m!$ ways to define $f(2), f(4), \ldots, f(2m)$. Similarly, there are $m + 1$ ways to define $f(1)$, m ways to define $f(3)$, and so on down to just one way to define $f(2m+1)$. So there are $(m+1)!$ ways to define $f(1), f(3), \ldots, f(2m + 1)$. Thus altogether there are $\mathbf{m!(m+1)!}$ ways to define f so that it is one-to-one and onto.

(c) This time we want to count the number of functions $f \in \mathcal{F}$ so that fRg. But since $g(k) = 1$ for all $k \in \{1, 2, \ldots, n\}$, to get $f(k) + g(k)$ to be even for all k, we will need that $f(k)$ is odd for all k. Since $n = 2m$, there are m odd numbers in $\{1, 2, \ldots, n\}$. So we have m choices for each $f(k)$, and thus the total number of ways we can define f is $m^n = \mathbf{m^{2m}}$.

If we insist that f be one-to-one and onto, once again we only need to make it one-to-one. But since $f(k)$ must be odd for every k, and the total number of k's ($2m$) is greater than the number of odd numbers available (m), it is impossible to assign a different odd number to each $f(k)$. Thus the number of one-to-one onto functions this time is **zero**.