MATH 271 ASSIGNMENT 2 SOLUTIONS

1. (a) Find all positive integers a so that $\lfloor a/271 \rfloor = 10$. How many such integers are there?
   (b) Find all positive integers a so that $\lfloor 271/a \rfloor = 10$.
   (c) Find all positive integers a so that $\lceil 271/a \rceil = 10$.
   (d) Prove or disprove: $\forall n \in \mathbf{Z}$, the equations $\lfloor 271/x \rfloor = n$ and $\lceil 271/x \rceil = n$ have the *same number* of integer solutions x.
   (e) Prove or disprove: $\exists n \in \mathbf{Z}$ so that $\forall a \in \mathbf{Z}$, $\lfloor 271/a \rfloor \neq n$.

   (a) For $\lfloor a/271 \rfloor = 10$ to be true we would need $10 \leq a/271 < 11$, or $2710 \leq a < 271 \cdot 11 = 2981$. So the values of a are $2710, 2711, 2712, \ldots, 2980$, a total of 271 integers.

   (b) Now we will need $10 \leq 271/a < 11$, or $10a \leq 271 < 11a$. This means $a \leq 271/10$ and $a > 271/11$, in other words $24.6 < a \leq 27.1$. So the allowed values of a are $25, 26, 27$.

   (c) Similarly, this time we will need $9 < 271/a \leq 10$, or $9a < 271 \leq 10a$. This means $a < 271/9$ and $a \geq 271/10$, in other words $27.1 \leq a < 30.1$. So the allowed values of a are $28, 29, 30$.

   (d) Despite the "evidence" from parts (b) and (c) (where there were 3 solutions each time), this statement is *false*. One counterexample is $n = 11$, as the only solutions for $\lfloor 271/x \rfloor = 11$ are $x = 23$ and 24, while $\lceil 271/x \rceil = 11$ has the three solutions $x = 25, 26$ and 27. Another counterexample is $n = 8$, since $\lfloor 271/x \rfloor = 8$ has three solutions $x = 31, 32$ and 33 while $\lceil 271/x \rceil = 8$ has the five solutions $x = 34$ to 38.

   An interesting counterexample is $n = 1$. Notice that the equation $\lceil 271/x \rceil = 1$ means $0 < 271/x \leq 1$, which is satisfied for *every* integer x greater than or equal to 271. So there are infinitely many solutions. But the equation $\lfloor 271/x \rfloor = 1$ means $1 \leq 271/x < 2$, and this inequality is satisfied only for the integers $x = 136, 137, \ldots, 271$.

   (e) This statement is *true*, and there are lots of integers n satisfying the condition. For example, any $n > 271$ will work, because $\lfloor 271/a \rfloor > 271$ is impossible for an integer a.

   Note. Can you find the *smallest* positive integer n for which this statement is true? If you think you have an answer to this question, talk to your professor or TA.

2. Let

$$S_n = \frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \frac{9}{4 \cdot 5} + \cdots - \frac{4n-3}{(2n-2)(2n-1)} + \frac{4n-1}{(2n-1)2n},$$

where the signs alternate.

   (a) Calculate and simplify S_1, S_2 and S_3.
   (b) Use part (a) (and more calculations if you need them) to guess a simple formula for S_n.
   (c) Prove your formula for all positive integers n *using mathematical induction*.
   (d) Give another proof of your formula for all positive integers n *using telescoping*. (See example 4.1.10 on page 205 of the text.)

(a) We get

$$S_1 = \frac{3}{1\cdot 2} = \frac{3}{2}\,, \quad S_2 = \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} = \frac{3}{2} - \frac{5}{6} + \frac{7}{12} = \frac{18-10+7}{12} = \frac{15}{12} = \frac{5}{4}\,,$$

and (using our calculation for S_2)

$$S_3 = \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \frac{11}{5\cdot 6} = \frac{5}{4} - \frac{9}{20} + \frac{11}{30} = \frac{75-27+22}{60} = \frac{70}{60} = \frac{7}{6}\,.$$

(b) From the values in (a) we guess that $S_n = \dfrac{2n+1}{2n}$.

(c) *Basis step.* We need to prove that $S_1 = \dfrac{2\cdot 1+1}{2\cdot 1}$, which is true since both are 3/2.

Induction step. Assume that $S_k = \dfrac{2k+1}{2k}$ for some integer $k \geq 1$. We want to prove that $S_{k+1} = \dfrac{2(k+1)+1}{2(k+1)}$, which is the same as $S_{k+1} = \dfrac{2k+3}{2(k+1)}$. Well,

$$\begin{aligned}
S_{k+1} &= \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \cdots + \frac{4k-1}{(2k-1)2k} - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \\
&= S_k - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \\
&= \frac{2k+1}{2k} - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \qquad \text{by our assumption} \\
&= \frac{(2k+1)^2(2k+2) - (4k+1)(2k+2) + (4k+3)2k}{2k(2k+1)(2k+2)} \\
&= \frac{[4k^2+4k+1-(4k+1)](2k+2) + (8k^2+6k)}{2k(2k+1)(2k+2)} \\
&= \frac{4k^2(2k+2) + (8k^2+6k)}{2k(2k+1)(2k+2)} = \frac{8k^3+16k^2+6k}{2k(2k+1)(2k+2)} \\
&= \frac{2k(2k+1)(2k+3)}{2k(2k+1)(2k+2)} = \frac{2k+3}{2k+2}\,,
\end{aligned}$$

which proves the induction step.

Therefore the statement is true for all integers $n \geq 1$.

(d) Notice that

$$\frac{3}{1\cdot 2} = \frac{1}{1} + \frac{1}{2}\,, \quad \frac{5}{2\cdot 3} = \frac{1}{2} + \frac{1}{3}\,, \quad \frac{7}{3\cdot 4} = \frac{1}{3} + \frac{1}{4}\,,$$

and in general

$$\frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$$

for any positive integer k. Thus

$$
\begin{aligned}
S_n &= \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \cdots - \frac{4n-3}{(2n-2)(2n-1)} + \frac{4n-1}{(2n-1)2n} \\
&= \left(\frac{1}{1}+\frac{1}{2}\right) - \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) - \cdots - \left(\frac{1}{2n-2}+\frac{1}{2n-1}\right) + \left(\frac{1}{2n-1}+\frac{1}{2n}\right) \\
&= \frac{1}{1} + \frac{1}{2n} = \frac{2n+1}{2n},
\end{aligned}
$$

so our guess is proved.

3. (a) Prove the following statement by contradiction: for all integers n, if $3|n$ then $3 \not| (n+271)$.

   (b) Prove or disprove: for all integers n, if $3|n$ then $5 \not| n$.

   (c) Prove by mathematical induction that $3|(2^n - (-1)^n)$ for all integers $n \geq 1$.

   (a) Assume that $3|n$ for some integer n. This means that $n = 3k$ for some integer k. We want to prove that $3 \not| (n+271)$. To get a proof by contradiction, we assume that what we want to prove is false: namely we will assume that $3|(n+271)$. This means we are also assuming that $n + 271 = 3\ell$ for some integer ℓ. Now our two assumptions tell us that

$$271 = (n+271) - n = 3\ell - 3k = 3(\ell - k),$$

   where $\ell - k$ is an integer. Thus $3|271$, which however is false. Thus our assumption that $3|(n+271)$ must be false, so $3 \not| (n+271)$.

   (b) This is *false*. A counterexample is $n = 15$, since $3|15$ but also $5|15$. Another counterexample is $n = 0$.

   (c) *Basis step.* We need to prove that $3|(2^1 - (-1)^1)$, which says $3|(2+1)$ or $3|3$. This is true.

   Induction step. Assume that $3|(2^k - (-1)^k)$ for some integer $k \geq 1$. This means that $2^k - (-1)^k = 3\ell$ for some integer ℓ. We want to prove that $3|(2^{k+1} - (-1)^{k+1})$. Well,

$$
\begin{aligned}
2^{k+1} - (-1)^{k+1} &= 2\cdot 2^k - (-1)\cdot(-1)^k \\
&= 2(2^k - (-1)^k) + 2(-1)^k + (-1)^k \\
&= 2(3\ell) + 3(-1)^k \quad \text{by our assumption} \\
&= 3(2\ell + (-1)^k),
\end{aligned}
$$

   where $2\ell + (-1)^k$ is an integer. Thus $3|(2^{k+1} - (-1)^{k+1})$.

   Therefore, by induction, $3|(2^n - (-1)^n)$ for all integers $n \geq 1$.