MATH 271 ASSIGNMENT 4

Due 4:00 PM THURSDAY, March 24, 2005. Put your assignment in the appropriate **wooden slot** (corresponding to your lecture section and last name) inside room MS 315. Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. For each positive integer n, let $[n] = \{1, 2, 3, \ldots, n\}$, and define

$$\begin{aligned} \mathcal{S}_{\cup}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of sets such that } A \cup B = [n]; \\ \mathcal{S}_{\cap}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \cap B = \emptyset; \\ \mathcal{S}_{\subseteq}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \subseteq B. \end{aligned}$$

   (a) Find $\mathcal{S}_{\cup}(1)$ and $\mathcal{S}_{\cup}(2)$.

   (b) Prove that $\mathcal{S}_{\cup}(n)$ has exactly 3^n elements.

   (c) Prove that $(A, B) \in \mathcal{S}_{\cup}(n)$ if and only if $(A^c, B^c) \in \mathcal{S}_{\cap}(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_{\cap}(n)$.

   (d) Prove that $(A, B) \in \mathcal{S}_{\cup}(n)$ if and only if $(A^c, B) \in \mathcal{S}_{\subseteq}(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_{\subseteq}(n)$.

2. For each positive integer n, let $f(n)$ be the number of ordered pairs (A, B) of subsets of $\{1, 2, 3, \ldots, n\}$ so that $A \cup B$ has an even number of elements.

   (a) Find $f(1)$ and $f(2)$ by listing all the ordered pairs of subsets.

   (b) Use Problem 1(b) to prove that for any n,

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k}.$$

   Show that your answers to part (a) agree with this formula.

   (c) Mimic Example 6.7.4 on page 368 to prove that $\sum_{i=0}^{n} \binom{n}{i} 3^i = 4^n$ and thus

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n - f(n).$$

   (d) Use Pascal's Formula (page 360), (b) and (c), and mathematical induction to prove that

$$f(n) = \begin{cases} 2^{n-1}(2^n - 1) & \text{if } n \text{ is odd,} \\ 2^{n-1}(2^n + 1) & \text{if } n \text{ is even.} \end{cases}$$

3. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer n.

   (a) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (b) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (c) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.

   (d) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.