MATH 271 ASSIGNMENT 3 SOLUTIONS

1. (a) Prove **by induction** that, for all integers $n \geq 2$,

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{n^2}{(n+1)!} \leq 2 - \frac{2n}{(n+1)!} \ . \tag{1}$$

(b) Prove that in fact inequality (1) holds for all integers $n \geq 1$.

(c) Find the smallest real number A so that, for all integers $n \geq 1$,

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{n^2}{(n+1)!} \leq A - \frac{2n}{(n+1)!} \ .$$

(a) *Basis step.* When $n = 2$ inequality (1) is

$$\frac{1^2}{2!} + \frac{2^2}{3!} \leq 2 - \frac{4}{3!}$$

which is

$$\frac{1}{2} + \frac{4}{6} \leq 2 - \frac{4}{6} \ , \quad \text{that is} \quad \frac{7}{6} \leq \frac{8}{6} \ ,$$

which is true.

Inductive step. Assume that inequality (1) holds for some integer $n = k$, where $k \geq 2$. We want to prove that inequality (1) holds for $n = k + 1$. So we are assuming that

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{k^2}{(k+1)!} \leq 2 - \frac{2k}{(k+1)!} \ ,$$

and we want to prove that

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{(k+1)^2}{(k+2)!} \leq 2 - \frac{2(k+1)}{(k+2)!} \ . \tag{2}$$

Well,

$$\begin{aligned}
\frac{1^2}{2!} + \frac{2^2}{3!} + \cdots + \frac{(k+1)^2}{(k+2)!} &= \frac{1^2}{2!} + \frac{2^2}{3!} + \cdots + \frac{k^2}{(k+1)!} + \frac{(k+1)^2}{(k+2)!} \\
&\leq 2 - \frac{2k}{(k+1)!} + \frac{(k+1)^2}{(k+2)!} \quad \text{by our assumption} \\
&= 2 - \frac{2k(k+2) - (k+1)^2}{(k+2)!} \\
&= 2 - \frac{2k^2 + 4k - k^2 - 2k - 1}{(k+2)!} \\
&= 2 - \frac{k^2 + 2k - 1}{(k+2)!} \ .
\end{aligned}$$

So in order to prove (2), we would like to prove that

$$2 - \frac{k^2+2k-1}{(k+2)!} \le 2 - \frac{2(k+1)}{(k+2)!} \ .$$

This is equivalent successively to

$$-\frac{k^2+2k-1}{(k+2)!} \le -\frac{2(k+1)}{(k+2)!} \ ,$$

$$\frac{k^2+2k-1}{(k+2)!} \ge \frac{2(k+1)}{(k+2)!} \ ,$$

and thus to

$$k^2+2k-1 \ge 2k+2, \quad \text{that is,} \quad k^2 \ge 3,$$

which is true since $k \ge 2$. This finishes the proof of the inductive step. Thus inequality (1) holds for all integers $n \ge 2$.

(b) When $n = 1$, inequality (1) says

$$\frac{1^2}{2!} \le 2 - \frac{2}{2!}$$

which is $1/2 \le 1$, which is true. Since in part (a) we proved that inequality (1) holds for all integers $n \ge 2$, we now know it holds for all integers $n \ge 1$. Notice that, since the inductive step needed that $k \ge 2$, to prove inequality (1) for all $n \ge 1$ we need both cases $n = 1$ and $n = 2$ in the basis step.

(c) The inductive step in the proof in part (a) works just the same if the 2 right after the inequality sign is replaced with any number A. So the inequality in part (c) will hold for all integers $n \ge 1$ provided that it holds for $n = 1$ and $n = 2$, which is the basis step. When $n = 1$ the inequality in (c) says

$$\frac{1^2}{2!} \le A - \frac{2}{2!}$$

which simplifies to $A \ge 3/2$. When $n = 2$ the inequality in (c) says

$$\frac{1^2}{2!} + \frac{2^2}{3!} \le A - \frac{4}{3!}$$

which simplifies to $A \ge 1/2 + 4/6 + 4/6 = 11/6$. We need both of these to hold, so the smallest A that will work is $A = \mathbf{11/6}$.

2. You are given the following "while" loop:

[*Pre-condition:* m is a nonnegative integer, $a = 0$, $b = 1$, $c = 2$, $i = 0$.]

while $(i \neq m)$

1. $a := b$
2. $b := c$
3. $c := 2b - a$
4. $i := i + 1$

end while

[*Post-condition:* $c = m + 2$.]

Loop invariant: $I(n)$ is "$a = n$, $b = n + 1$, $c = n + 2$, $i = n$".

(a) Prove the correctness of this loop with respect to the pre- and post-conditions.

(b) Suppose the "while" loop is as above, except that the pre-condition is replaced by: m is a nonnegative integer, $a = 1$, $b = 3$, $c = 5$, $i = 0$. Find a post-condition that gives the final value of c, and an appropriate loop invariant, and prove the correctness of this loop.

(a) We first need to check that the loop invariant holds when $n = 0$. $I(0)$ says $a = 0$, $b = 1$, $c = 2$ and $i = 0$, and these are all true by the pre-conditions.

So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$, $k < m$. We want to prove that $I(k + 1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that $a = k$, $b = k + 1$, $c = k + 2$ and $i = k$, and we now go through the loop. Step 1 sets a equal to $b = k + 1$, then step 2 sets b equal to $c = k + 2$, then step 3 sets c equal to $2b - a = 2(k + 2) - (k + 1) = k + 3$, then step 4 sets i equal to $k + 1$. This means that $I(k + 1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $c = m + 2$ as required.

(b) If we set the variables to their pre-condition values of $a = 1$, $b = 3$, $c = 5$ and $i = 0$, and run through the loop, the new values we get are $a = 3$, $b = 5$, $c = 2(5) - 3 = 7$, and $i = 1$. From this (or by running through the loop once or twice more to collect more evidence) we can guess that the loop invariant we want will be

$$I(n) : a = 2n + 1, b = 2n + 3, c = 2n + 5, i = n,$$

and the post-condition value of c ought to be $c = 2m + 5$. This choice of $I(n)$ becomes $a = 1$, $b = 3$, $c = 5$ and $i = 0$ when $n = 0$, so the pre-condition is satisfied.

So now we assume that the new loop invariant $I(k)$ holds for some integer $k \geq 0$, $k < m$, and we want to prove that $I(k + 1)$ holds. So we are assuming that $a = 2k + 1$, $b = 2k + 3$, $c = 2k + 5$ and $i = k$, and we now go through the loop. Step 1 sets a equal to $b = 2k + 3 = 2(k + 1) + 1$, then step 2 sets b equal to $c = 2k + 5 = 2(k + 1) + 3$, then step 3 sets c equal to $2b - a = 2(2k + 5) - (2k + 3) = 2k + 7 = 2(k + 1) + 5$, then step 4 sets i equal to $k + 1$. This means that $I(k + 1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $c = 2m + 5$ as required.

3. Prove or disprove each of the following six statements. Proofs should use the "element" methods given in Section 5.2. [*Note*: $\mathcal{P}(X)$ denotes the power set of the set X.]

   (a) For all sets A, B, C, $(A - B) \times C \subseteq (A \times C) - (B \times C)$.

   (b) For all sets A, B, C, $(A \times C) - (B \times C) \subseteq (A - B) \times C$.

   (c) For all sets A, B, C, $(A - B) \times C = (A \times C) - (B \times C)$.

   (d) For all sets A and B, $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   (e) For all sets A and B, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   (f) For all sets A and B, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

   (a) This inequality is **true**. Here is a proof.

   Let A, B, C be arbitrary sets. Note that the left side of this inequality is a Cartesian product, which means that its elements will be ordered pairs. So let (a, c) be an arbitrary element of $(A - B) \times C$. This means that $a \in A - B$ and $c \in C$. Since $a \in A - B$, this means that $a \in A$ and $a \notin B$. Since $a \in A$ and $c \in C$, we get that $(a, c) \in A \times C$. But since $a \notin B$, we know that (a, c) cannot be an element of $B \times C$. Since $(a, c) \in A \times C$ but $(a, c) \notin B \times C$, we know $(a, c) \in (A \times C) - (B \times C)$. Therefore $(A - B) \times C \subseteq (A \times C) - (B \times C)$.

   (b) Similarly, this inequality is **true**, and we can reverse our steps in part (a) to get a proof. Let (a, c) be an arbitrary element of $(A \times C) - (B \times C)$. This means that $(a, c) \in A \times C$ but $(a, c) \notin B \times C$. Since $(a, c) \in A \times C$, we know that $a \in A$ and $c \in C$. But since $(a, c) \notin B \times C$ although $c \in C$, we also know $a \notin B$. Thus $a \in A$ and $a \notin B$, which means $a \in A - B$. Thus $(a, c) \in (A - B) \times C$. Therefore $(A \times C) - (B \times C) \subseteq (A - B) \times C$.

   (c) Since the inequalities in parts (a) and (b) both hold, we get that the equality in (c) holds for all sets A, B, C.

   (d) This inequality is **false** no matter what sets we choose for A and B! To see this, let A and B be any sets. Notice that the empty set $\emptyset \subseteq A - B$ regardless of what A and B are, so $\emptyset \in \mathcal{P}(A - B)$. However, since $\emptyset \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(B)$, we get $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$. Therefore $\mathcal{P}(A - B) \not\subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   Note. You can prove that if X is any *nonempty* set so that $X \in \mathcal{P}(A - B)$, then $X \in \mathcal{P}(A) - \mathcal{P}(B)$. So the only counterexample to the inequality in part (d) is the empty set.

   (e) This inequality is also **false**, but counterexamples are easier to find. For example, let $A = \{1, 2\}$ and $B = \{1\}$. Then $\{1, 2\} \subseteq A$ and $\{1, 2\} \not\subseteq B$, so $\{1, 2\} \in \mathcal{P}(A)$ and $\{1, 2\} \notin \mathcal{P}(B)$, so $\{1, 2\} \in \mathcal{P}(A) - \mathcal{P}(B)$. However $A - B = \{2\}$, so $\{1, 2\} \notin \mathcal{P}(A - B)$. Therefore $\mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B)$.

   (f) Since the inequality in (e) (or (d)) fails, the equality in (f) fails too.