MATH 271 ASSIGNMENT 3 SOLUTIONS

1. Let n be a positive integer. If $A_1, A_2, \ldots, A_n$ are sets, we write

$$\mathcal{S}_n = A_1 - (A_2 - (A_3 - (\cdots - (A_{n-1} - A_n))\cdots)).$$

For example, if $n = 4$ then $\mathcal{S}_4 = A_1 - (A_2 - (A_3 - A_4))$.

(a) Let A be a set, and let $A_i = A$ for all $1 \le i \le n$, so that

$$\mathcal{S}_n = A - (A - (A - (\cdots - (A - A))\cdots)),$$

where there are n A's. Prove (using induction or well ordering) that

$$\mathcal{S}_n = \begin{cases} A & \text{if } n \text{ is odd} \\ \emptyset & \text{if } n \text{ is even.} \end{cases}$$

(b) Prove that for all sets A and B, $A - (B - A) = A$. You may use the identities on page 272.

(c) Let A and B be sets, and let $A_i = \begin{cases} A & \text{if } i \text{ is odd} \\ B & \text{if } i \text{ is even} \end{cases}$. Find a simple formula (something like in part (a)) for $\mathcal{S}_n$, and prove it using induction or well ordering.

(a) *Basis Step.* If $n = 1$, then $\mathcal{S}_1 = A$, which agrees with what we want since 1 is odd.

Inductive Step. Assume that the formula for $\mathcal{S}_k$ is correct for some positive integer k. Then

$$\begin{aligned} \mathcal{S}_{k+1} &= A - (A - (A - (\cdots - (A - A))\cdots)) \quad (\text{with } k+1 \text{ } A\text{'s}) \\ &= A - \mathcal{S}_k \\ &= \begin{cases} A - A & \text{if } k \text{ is odd} \\ A - \emptyset & \text{if } k \text{ is even} \end{cases} \quad \text{by assumption} \\ &= \begin{cases} \emptyset & \text{if } k+1 \text{ is even} \\ A & \text{if } k+1 \text{ is odd} \end{cases} \end{aligned}$$

so the formula for $\mathcal{S}_{k+1}$ is correct.

Therefore, by induction, the formula for $\mathcal{S}_n$ is correct for all positive integers n.

(b) Using the identities on page 272,

$$\begin{aligned} A - (B - A) &= A - (B \cap A^c) = A \cap (B \cap A^c)^c \quad (\#12) \\ &= A \cap (B^c \cup (A^c)^c) \quad (\#9) \\ &= A \cap (B^c \cup A) \quad (\#6) \\ &= (A \cap B^c) \cup (A \cap A) \quad (\#3) \\ &= (A \cap B^c) \cup A \quad (\#7) \\ &= A. \quad (\#1, \#10) \end{aligned}$$

You could also prove this using the element method:

To prove $A-(B-A)\subseteq A$: Let $x\in A-(B-A)$ be arbitrary. This implies $x\in A$, so $A-(B-A)\subseteq A$.

To prove $A\subseteq A-(B-A)$: Let $x\in A$ be arbitrary. Then notice that $x\notin B-A$, because if $x\in B-A$ it implies $x\notin A$ which is a contradiction. Since $x\notin B-A$, we get $x\in A-(B-A)$. Thus $A\subseteq A-(B-A)$.

Therefore $A-(B-A)=A$.

(c) We get $\mathcal{S}_1=A$, $\mathcal{S}_2=A-B$, and $\mathcal{S}_3=A-(B-A)$ which is equal to A by part (b). So we guess that

$$\mathcal{S}_n=\begin{cases} A & \text{if } n \text{ is odd}\\ A-B & \text{if } n \text{ is even,}\end{cases}$$

and we now prove this by induction.

Basis Step. If $n=1$, then $\mathcal{S}_1=A$, which agrees with our guess since 1 is odd.

Inductive Step. Assume that our guessed formula for $\mathcal{S}_k$ is correct for some positive integer k. Then

$$\mathcal{S}_{k+1}=\begin{cases} A-(B-(A-(\cdots-(B-A))\cdots)) & \text{if } k+1 \text{ is odd}\\ A-(B-(A-(\cdots-(A-B))\cdots)) & \text{if } k+1 \text{ is even.}\end{cases}$$

Note that the expression $B-(A-(\cdots-(B-A))\cdots)$ (or $B-(A-(\cdots-(A-B))\cdots))$ inside the outer parentheses is just $\mathcal{S}_k$ with A and B switched. Thus by assumption this expression must be

$$\begin{cases} B & \text{if } k \text{ is odd}\\ B-A & \text{if } k \text{ is even.}\end{cases}$$

Therefore

$$\mathcal{S}_{k+1}=\begin{cases} A-(B-A)=A & \text{if } k+1 \text{ is odd} \quad \text{(again using part (b))}\\ A-B & \text{if } k+1 \text{ is even,}\end{cases}$$

so the formula for $\mathcal{S}_{k+1}$ is correct.

Therefore, by induction, the formula for $\mathcal{S}_n$ is correct for all positive integers n.

2. There are 5 men and 5 women, of 10 different heights.

(a) Find the number of ways of arranging the 10 people in a row so that the ith shortest woman is next to the ith shortest man, for all $1\leq i\leq 5$.

(b) Find the number of ways of arranging the 10 people in a row so that the women occupy five consecutive spots.

(c) Find the number of ways of arranging the 10 people in a row so that everyone except the tallest person is next to someone taller.

(a) Since the ith shortest man and the ith shortest woman must be next to each other for each i, we can think of each couple as being "tied together" and first arrange the five couples. This can be done in 5! ways. For each such arrangement, we can put each couple in two orders (MW or WM), which doubles the number of arrangements for each couple. Thus the total number of arrangements is $5! \cdot 2^5 = 120 \cdot 32 = 3840$.

(b) The five consecutive spots the women occupy could be in 6 different locations: spots 1 to 5, 2 to 6, 3 to 7, 4 to 8, 5 to 9, or 6 to 10. For each of these choices, there are 5! ways to arrange the women in these spots, and for each such way of arranging the women there are also 5! ways of arranging the men in the five remaining spots. Thus there are $6 \cdot 5! \cdot 5! = 6(120)^2 = 86400$ such arrangements.

(c) Suppose the people are A_1 to A_{10} from tallest to shortest. To make such an arrangement, start with A_1. Then A_2 must stand next to A_1, so there are two choices for where A_2 goes, either to the left or the right of A_1. Then A_3 must stand next to either A_1 or A_2, but not in between them, so A_3 must go at one end of the "line" formed by A_1 and A_2, so there are two choices for where A_3 goes. Then A_4 must go at one end of the line formed by A_1, A_2 and A_3, so there are two choices for A_4. And so on, considering each person in order from tallest to shortest, each person must go at one end of the line formed by all the taller people. Thus each of the 9 people other than the tallest person has two choices for where to be in line, so there are $2^9 = 512$ arrangements altogether.

Another way to do this problem is by induction. First we would need to get a guess for the correct answer for the general problem where there are n people of different heights. In fact we may as well rephrase the problem to be: find the number of arrangements of the numbers $1, 2, \ldots, n$ so that each number except n is next to a larger number. We'll call these "good" arrangements. For small numbers n you can count the good arrangements: for $n = 2$ there are two (12 and 21), for $n = 3$ there are four (123, 132, 231, 321), and so on. Soon we get the guess that there should be 2^{n-1} good arrangements of the numbers $1, 2, \ldots, n$, and we now can prove this by induction. We already know the formula is correct for $n = 2$. So suppose that the formula is correct for some integer $n = k$, where $k \geq 2$. So there are 2^{k-1} good arrangements of the numbers $1, 2, \ldots, k$. Notice that for each such arrangement, we can stick the number $k + 1$ in on either side of the number k (wherever it is), and we will get a good arrangement of $1, 2, \ldots, k + 1$, because now k is next to a larger number, and any number that used to be next to k either still is or else is now next to the even larger number $k+1$. This gives us $2 \cdot 2^{k-1} = 2^k$ good arrangements of $1, 2, \ldots, k + 1$. Moreover each good arrangement of $1, 2, \ldots, k + 1$ will get counted this way, because if you take a good arrangement of $1, 2, \ldots, k + 1$ and pull out the $k + 1$ (and close up the gap) you will get a good arrangement of $1, 2, \ldots, k$, because we know k had to be next to $k + 1$, so any other number ℓ that used to be next to $k+1$ will now be next to k, and $\ell < k$, so everything is still okay. Thus there are exactly 2^k good arrangements of $1, 2, \ldots, k + 1$, which agrees with the formula when $n = k + 1$. This finishes the Inductive Step, and so we know that there are exactly 2^{n-1} good arrangements of $1, 2, \ldots, n$ for each integer $n \geq 1$. In particular, there are 2^9 good arrangements when $n = 10$, which answers the question.

3. Find the number of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 10\}$ satisfying:

   (a) $N(A \cap B) = 7$. ($N(X)$ is the number of elements in the set X; see page 299.)

   (b) $N(A \times B) = 7$.

   (c) $N\big(\mathcal{P}(A \cup B)\big) = 7$. ($\mathcal{P}(X)$ is the power set of the set X.)

   (d) $N\big(\mathcal{P}(A) \cup \mathcal{P}(B)\big) = 7$.

   (a) We want $A \cap B$ to have exactly 7 elements from the set $\{1, 2, \ldots, 10\}$. There are $\binom{10}{7}$ ways to choose these 7 elements. For each such choice, the other three elements in $\{1, 2, \ldots, 10\}$ could be in either A or B (or neither), but cannot be in both (or the size of the intersection would be too large). So each of the 3 other elements has three choices: in A, in B, or in neither. This means there are $3^3 = 27$ ways to distribute the 3 elements not in $A \cap B$, for each of the $\binom{10}{7}$ choices for $A \cap B$, so there are $27\binom{10}{7} = 27\binom{10}{3} = \frac{27 \cdot 10 \cdot 9 \cdot 8}{3 \cdot 2} = 3240$ such choices of ordered pairs (A, B) altogether.

   (b) Since $7 = N(A \times B) = N(A) \times N(B)$ and 7 is prime, we would need either $N(A) = 7$ and $N(B) = 1$, or $N(A) = 1$ and $N(B) = 7$. The number of 7-element subsets of $\{1, 2, \ldots, 10\}$ is $\binom{10}{7} = \binom{10}{3} = 120$, and the number of 1-element subsets is $\binom{10}{1}$ which of course is 10. So the number of ways to choose A and B with $N(A) = 7$ and $N(B) = 1$ is $120 \times 10 = 1200$, and the number of ways to choose A and B with $N(A) = 1$ and $N(B) = 7$ is also 1200. So there are 2400 such ordered pairs (A, B) altogether.

   (c) For any set X with n elements, its power set $\mathcal{P}(X)$ has 2^n elements. Since 7 is not a power of 2, $\mathcal{P}(A \cup B)$ cannot be equal to 7, so there are 0 (zero) such ordered pairs (A, B) in this case.

   (d) We want $\mathcal{P}(A) \cup \mathcal{P}(B)$ to have exactly 7 elements, where the number of elements in each of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ separately must be a power of 2. If either A or B had three or more elements, then their power set would already have at least 8 elements, which is too big. On the other hand, if say A had only one element, then its power set would have only two elements, while $\mathcal{P}(B)$ would have at most 4 elements, which totals to at most 6 elements, not enough. So each of A and B must have exactly two elements. Then they both have $2^2 = 4$ subsets, but notice that the empty set is a subset of each, which means that $\mathcal{P}(A) \cup \mathcal{P}(B)$ would have at most $4 + 4 - 1 = 7$ elements. If A and B had any elements in common, the number of subsets in $\mathcal{P}(A) \cup \mathcal{P}(B)$ would be even smaller, so it means that what we want to count is the number of ordered pairs (A, B) of disjoint 2-element subsets of $\{1, 2, \ldots, 10\}$. The number of choices for A is $\binom{10}{2} = 45$, and for each choice of A the number of choices for B is $\binom{8}{2} = 28$. So the total number of ordered pairs (A, B) is $45 \cdot 28 = 1260$.

   You could also do this count by first counting the number of ways to choose the four elements in $A \cup B$ (which is $\binom{10}{4} = 210$), and multiplying by the number of ways to choose two of these four elements to be A (which is $\binom{4}{2} = 6$), getting $210 \cdot 6 = 1260$ ways to choose A and B.