MATH 271 ASSIGNMENT 2 SOLUTIONS

1. For an integer $n \geq 1$, let $S(n)$ be the statement

$$2 + \frac{1}{24} - \frac{2}{n+1} \leq \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2n-1}{n^3} \leq 3 - \frac{2}{n} \ .$$

   (a) Prove **by induction** (or by well-ordering) that $S(n)$ is true for all integers $n \geq 2$.

   (b) Let N be your student ID number. Use (a) to find

$$\left\lfloor \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2N-1}{N^3} \right\rfloor .$$

   (a) *Basis step.* When $n = 2$ $S(2)$ is

$$2 + \frac{1}{24} - \frac{2}{3} \leq \frac{1}{1^3} + \frac{3}{2^3} \leq 3 - \frac{2}{2}$$

which is

$$\frac{11}{8} \leq 1 + \frac{3}{8} \leq 2,$$

which is true.

Inductive step. Assume that $S(k)$ holds for some integer $k \geq 2$. We want to prove that $S(k+1)$ holds. So we are assuming that

$$2 + \frac{1}{24} - \frac{2}{k+1} \leq \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2k-1}{k^3} \leq 3 - \frac{2}{k} \ , \tag{1}$$

and we want to prove that

$$2 + \frac{1}{24} - \frac{2}{k+2} \leq \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2(k+1)-1}{(k+1)^3} \leq 3 - \frac{2}{k+1} \ . \tag{2}$$

From (1) we get

$$2 + \frac{1}{24} - \frac{2}{k+1} + \frac{2k+1}{(k+1)^3} \leq \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2k-1}{k^3} + \frac{2k+1}{(k+1)^3} \leq 3 - \frac{2}{k} + \frac{2k+1}{(k+1)^3} \ .$$

So in order to prove (2), we would like to prove that

$$2 + \frac{1}{24} - \frac{2}{k+2} \leq 2 + \frac{1}{24} - \frac{2}{k+1} + \frac{2k+1}{(k+1)^3} \tag{3}$$

and

$$3 - \frac{2}{k} + \frac{2k+1}{(k+1)^3} \leq 3 - \frac{2}{k+1} \ . \tag{4}$$

Well,

$$\begin{aligned}
(3) \quad &\Longleftrightarrow \quad \frac{2}{k+1} - \frac{2}{k+2} \leq \frac{2k+1}{(k+1)^3} \\
&\Longleftrightarrow \quad \frac{2}{(k+1)(k+2)} \leq \frac{2k+1}{(k+1)^3} \\
&\Longleftrightarrow \quad 2(k+1)^2 \leq (k+2)(2k+1) \\
&\Longleftrightarrow \quad 2k^2 + 4k + 2 \leq 2k^2 + 5k + 2,
\end{aligned}$$

which is true for all integers $k \geq 2$. Thus (3) is true. Also,

$$\begin{aligned}
(4) \quad &\iff \frac{2k+1}{(k+1)^3} \leq \frac{2}{k} - \frac{2}{k+1} \\
&\iff \frac{2k+1}{(k+1)^3} \leq \frac{2}{k(k+1)} \\
&\iff (2k+1)k \leq 2(k+1)^2 \\
&\iff 2k^2 + k \leq 2k^2 + 4k + 2,
\end{aligned}$$

which is also true for all integers $k \geq 2$. Thus (4) is true too. This finishes the proof of the inductive step. Thus $S(n)$ holds for all integers $n \geq 2$.

(b) Since your student ID number N is greater than 47, $\frac{1}{24} > \frac{2}{N+1}$. Thus from (a),

$$2 < 2 + \frac{1}{24} - \frac{2}{N+1} \leq \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2N-1}{N^3} \leq 3 - \frac{2}{N} < 3.$$

Therefore

$$\left\lfloor \frac{1}{1^3} + \frac{3}{2^3} + \frac{5}{3^3} + \cdots + \frac{2N-1}{N^3} \right\rfloor = 2.$$

2. The sequence $b_0, b_1, b_2, \ldots$ is defined by: $b_0 = 1$, $b_1 = 1$, $b_2 = 6$, and $b_n = 3b_{n-2} + 2b_{n-3}$ for all integers $n \geq 3$.

(a) Find b_3, b_4 and b_5.

(b) Use part (a) (and more data if you need it) to guess a simple formula for b_n.
[*Hint*: how far away is b_4 from the *nearest* power of 2? How about b_5?]

(c) Use **strong induction** (or well-ordering) to prove your guess.

(a) We get

$$\begin{aligned}
b_3 &= 3b_1 + 2b_0 = 3 \cdot 1 + 2 \cdot 1 = 5, \\
b_4 &= 3b_2 + 2b_1 = 3 \cdot 6 + 2 \cdot 1 = 20, \\
b_5 &= 3b_3 + 2b_2 = 3 \cdot 5 + 2 \cdot 6 = 27.
\end{aligned}$$

(b) The nearest power of 2 to $b_4 = 20$ is $2^4 = 16$, which is 4 less than b_4. The nearest power of 2 to $b_5 = 27$ is $2^5 = 32$, which is 5 more than b_5. We could also find that $b_6 = 3b_4 + 2b_3 = 3 \cdot 20 + 2 \cdot 5 = 70$, which is 6 more than the nearest power of 2 which is $2^6 = 64$. So we guess that

$$b_n = \begin{cases} 2^n + n & \text{if } n \text{ is even,} \\ 2^n - n & \text{if } n \text{ is odd,} \end{cases}$$

which could also be written as: $b_n = 2^n + (-1)^n n$ for all integers $n \geq 0$.

(c) *Basis Step*. We must prove that our guessed formula for b_n is true when $n = 0, 1$ and 2. Note that $2^0 + (-1)^0 0 = 1 + 0 = 1 = b_0$, $2^1 + (-1)^1 1 = 2 - 1 = 1 = b_1$, and $2^2 + (-1)^2 2 = 4 + 2 = 6 = b_2$, so this all checks.

Inductive Step. Assume that the guessed formula is correct for all integers n between 0 and k inclusive, where $k \geq 2$ is some integer. We want to prove that the formula is correct for $n = k+1$, that is we want to prove that $b_{k+1} = 2^{k+1} + (-1)^{k+1}(k+1)$. Well,

$$\begin{aligned} b_{k+1} &= 3b_{k-1} + 2b_{k-2} \qquad \text{by the recurrence} \\ &= 3[2^{k-1} + (-1)^{k-1}(k-1)] + 2[2^{k-2} + (-1)^{k-2}(k-2)] \qquad \text{by assumption} \\ &= 3 \cdot 2^{k-1} + 3(-1)^{k-1}(-1)^2(k-1) + 2^{k-1} + 2(-1)^{k-2}(-1)^4(k-2) \\ &= 4 \cdot 2^{k-1} + (-1)^{k+1}[3(k-1) - 2(k-2)] \\ &= 2^{k+1} + (-1)^{k+1}(3k - 3 - 2k + 4) = 2^{k+1} + (-1)^{k+1}(k+1), \end{aligned}$$

so the formula is correct for $n = k+1$. Therefore the formula is correct for all integers $n \geq 0$.

3. You are given the following "while" loop:

   [*Pre-condition*: m is a nonnegative *even* integer, $a = 0$, $b = 0$, $c = 0$.]

   while $(a \neq m)$
   1. $b := 2a - b$
   2. $c := 2b - c$
   3. $a := a + 1$

   end while

   [*Post-condition*: $c = 2m$.]

   Loop invariant: $I(n)$ is

$$a = n, \quad b = \left\{ \begin{array}{ll} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{array} \right\}, \quad c = \left\{ \begin{array}{ll} 2n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array} \right\}.$$

   (a) Prove the correctness of this loop with respect to the pre- and post-conditions.

   (b) Suppose the "while" loop is as above, but $c = 1$ in the pre-condition, and statement 2 in the "while" loop is replaced by: $c := 2b - a$. Find a post-condition that gives the final value of c, and an appropriate loop invariant, and prove the correctness of this loop.

   (a) We first need to check that the loop invariant holds when $n = 0$. Since 0 is even, $I(0)$ says $a = 0$, $b = 0$ and $c = 2 \cdot 0 = 0$, and these are all true by the pre-conditions.

   So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$ where $k < m$. We want to prove that $I(k+1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that

$$a = k,\ b = k,\ c = 2k \quad \text{if } k \text{ is even}, \qquad a = k,\ b = k-1,\ c = 0 \quad \text{if } k \text{ is odd},$$

   and we now go through the loop.

   - Step 1: $b := 2a - b = \left\{ \begin{array}{ll} 2k - k = k & \text{if } k \text{ is even} \\ 2k - (k-1) = k+1 & \text{if } k \text{ is odd} \end{array} \right\}$
     $= \left\{ \begin{array}{ll} (k+1) - 1 & \text{if } k+1 \text{ is odd} \\ k+1 & \text{if } k+1 \text{ is even} \end{array} \right\}.$

- Step 2: $c := 2b - c = \left\{ \begin{array}{ll} 2k - 2k = 0 & \text{if } k \text{ is even} \\ 2(k+1) - 0 = 2(k+1) & \text{if } k \text{ is odd} \end{array} \right\}$

$$= \left\{ \begin{array}{ll} 0 & \text{if } k+1 \text{ is odd} \\ 2(k+1) & \text{if } k+1 \text{ is even} \end{array} \right\}.$$

- Step 3: $a := a + 1 = k + 1$.

This means that $I(k+1)$ is true, as required.

Finally the loop stops when $a = m$, and we need to check that at that point the post-condition is satisfied. When $a = m$ it means that the loop invariant $I(m)$ must hold, so, since m is even, from $I(m)$ we know that $c = 2m$ as required.

(b) If we set the variables to their pre-condition values of $a = 0$, $b = 0$ and $c = 1$, and run through the loop, the new values we get are $b = 2(0) - 0 = 0$, $c = 2(0) - 0 = 0$, $a = 1$. If we continue to run through the loop, and keep track of the variables in a table, here is what we get:

n	0	1	2	3	4	5
b	0	0	2	2	4	4
c	1	0	3	2	5	4
a	0	1	2	3	4	5

From this (or by running through the loop once or twice more to collect more evidence) we can guess that the loop invariant we want will be

$$I(n): \quad a = n, \quad b = \left\{ \begin{array}{ll} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{array} \right\}, \quad c = \left\{ \begin{array}{ll} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{array} \right\},$$

and the post-condition value of c ought to be $c = m + 1$, since m is even. This choice of $I(n)$ becomes $a = 0$, $b = 0$ and $c = 1$ when $n = 0$, so the pre-condition is satisfied.

So now we assume that the new loop invariant $I(k)$ holds for some integer $k \geq 0$, $k < m$, and we want to prove that $I(k+1)$ holds. So we are assuming that

$$a = k,\ b = k,\ c = k+1 \quad \text{if } k \text{ is even}, \qquad a = k,\ b = k-1,\ c = k-1 \quad \text{if } k \text{ is odd},$$

and we now go through the loop.

- Step 1: $b := 2a - b = \left\{ \begin{array}{ll} 2k - k = k & \text{if } k \text{ is even} \\ 2k - (k-1) = k+1 & \text{if } k \text{ is odd} \end{array} \right\}$

$$= \left\{ \begin{array}{ll} (k+1) - 1 & \text{if } k+1 \text{ is odd} \\ k+1 & \text{if } k+1 \text{ is even} \end{array} \right\}.$$

- Step 2: $c := 2b - a = \left\{ \begin{array}{ll} 2k - k = k & \text{if } k \text{ is even} \\ 2(k+1) - k = k+2 & \text{if } k \text{ is odd} \end{array} \right\}$

$$= \left\{ \begin{array}{ll} (k+1) - 1 & \text{if } k+1 \text{ is odd} \\ (k+1) + 1 & \text{if } k+1 \text{ is even} \end{array} \right\}.$$

- Step 3: $a := a + 1 = k + 1$.

This means that $I(k+1)$ is true, as required.

Finally the loop stops when $a = m$, and we need to check that at that point the post-condition is satisfied. When $a = m$ it means that the loop invariant $I(m)$ must hold, so, since m is even, from $I(m)$ we know that $c = m + 1$ as required.