MATH 271 ASSIGNMENT 4

Due 4:00 PM Thursday, April 5, 2007. Put your assignment (stapled, please) in the appropriate **wooden slot** (corresponding to your lecture section and last name) inside room MS 315. Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. If $F : X \to X$ is a function, define $f^2(x)$ to be $(f \circ f)(x)$, and inductively define $f^k(x) = (f \circ f^{k-1})(x)$ for each integer $k \geq 3$. (So $f^3(x) = (f \circ f^2)(x) = f(f(f(x)))$ for instance.) We also define $f^1(x)$ to be $f(x)$.

   Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by: for all $n \in \mathbb{Z}$, $f(n) = \begin{cases} 2-2n & \text{if } n \text{ is odd,} \\ 1-2n & \text{if } n \text{ is even.} \end{cases}$

   (a) Find $f^2(n)$, $f^3(n)$, and $f^4(n)$.

   (b) Use part (a) (and more data if you need it) to guess a fairly simple formula for $f^k(n)$ for any positive integer k. (You may need to consider k odd and k even separately.)

   (c) Use induction on k (or well ordering) to prove your guess.

2. For each integer $n \geq 3$, let G_n be the graph with vertex set $V(G_n) = \{1, 2, 3, \ldots, n\}$ and where, for all distinct $a, b \in V(G_n)$, ab is an edge if and only if $\gcd(a, b) = 1$.

   (a) Draw G_3 and G_4.

   (b) Find all integers $n \geq 3$ so that G_n has a Hamiltonian circuit.

   (c) Show that G_n does not have an Euler circuit if $n \bmod 4 \neq 3$ ($n \not\equiv 3 \bmod 4$). [*Hint*: do even n and odd n separately.]

   (d) Suppose for each integer $n \geq 3$ we define the graph G'_n the same way as for G_n except that $V(G'_n) = \{2, 3, \ldots, n\}$. Show (without a computer) that G'_8 does *not* have a Hamiltonian circuit. [*Hint*: start by thinking how 6 could fit into a Hamiltonian circuit.]

3. (a) Find the **total** number of all walks (starting at any vertex, ending at any vertex) of length 271 in the complete graph K_n.

   (b) Find the total number of walks of length 271 in the complete bipartite graph $K_{m,n}$.

   (c) Find the total number of *simple* paths of length $n-1$ in K_n.

   (d) Find the total number of simple paths of length $m+n-1$ in $K_{m,n}$. [You may assume that $m \geq n$. You will need to consider a few cases.]