

1. If $F : X \rightarrow X$ is a function, define $f^2(x)$ to be $(f \circ f)(x)$, and inductively define $f^k(x) = (f \circ f^{k-1})(x)$ for each integer $k \geq 3$. (So $f^3(x) = (f \circ f^2)(x) = f(f(f(x)))$ for instance.) We also define $f^1(x)$ to be $f(x)$.

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by: for all $n \in \mathbb{Z}$, $f(n) = \begin{cases} 2 - 2n & \text{if } n \text{ is odd,} \\ 1 - 2n & \text{if } n \text{ is even.} \end{cases}$

- (a) Find $f^2(n)$, $f^3(n)$, and $f^4(n)$.
 (b) Use part (a) (and more data if you need it) to guess a fairly simple formula for $f^k(n)$ for any positive integer k . (You may need to consider k odd and k even separately.)
 (c) Use induction on k (or well ordering) to prove your guess.
- (a) We get

$$\begin{aligned} f^2(n) &= f(f(n)) = \begin{cases} f(2 - 2n) & \text{if } n \text{ is odd,} \\ f(1 - 2n) & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} 1 - 2(2 - 2n) & \text{if } n \text{ is odd (since } 2 - 2n \text{ is even),} \\ 2 - 2(1 - 2n) & \text{if } n \text{ is even (since } 1 - 2n \text{ is odd).} \end{cases} \\ &= \begin{cases} 4n - 3 & \text{if } n \text{ is odd,} \\ 4n & \text{if } n \text{ is even;} \end{cases} \end{aligned}$$

$$\begin{aligned} f^3(n) &= f(f^2(n)) = \begin{cases} f(4n - 3) & \text{if } n \text{ is odd,} \\ f(4n) & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} 2 - 2(4n - 3) & \text{if } n \text{ is odd (since } 4n - 3 \text{ is odd),} \\ 1 - 2(4n) & \text{if } n \text{ is even (since } 4n \text{ is even).} \end{cases} \\ &= \begin{cases} 8 - 8n & \text{if } n \text{ is odd,} \\ 1 - 8n & \text{if } n \text{ is even;} \end{cases} \end{aligned}$$

$$\begin{aligned} f^4(n) &= f(f^3(n)) = \begin{cases} f(8 - 8n) & \text{if } n \text{ is odd,} \\ f(1 - 8n) & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} 1 - 2(8 - 8n) & \text{if } n \text{ is odd (since } 8 - 8n \text{ is even),} \\ 2 - 2(1 - 8n) & \text{if } n \text{ is even (since } 1 - 8n \text{ is odd).} \end{cases} \\ &= \begin{cases} 16n - 15 & \text{if } n \text{ is odd,} \\ 16n & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

- (b) From part (a) we might guess that if k is odd, then $f^k(n) = \begin{cases} 2^k - 2^k n & \text{if } n \text{ is odd,} \\ 1 - 2^k n & \text{if } n \text{ is even,} \end{cases}$

$$\text{and if } k \text{ is even, then } f^k(n) = \begin{cases} 2^k n - 2^k + 1 & \text{if } n \text{ is odd,} \\ 2^k n & \text{if } n \text{ is even.} \end{cases}$$

(c) *Basis step.* Our guessed formulas for $f^k(n)$ are true for $k = 1, 2, 3$ and 4 , by part (a).

Inductive step. Assume that our guessed formula is true for some integer $k = \ell \geq 1$. We want to prove that our formula is true when $k = \ell + 1$. We do this in two cases:

Case (i): ℓ is even. So we assume that $f^\ell(n) = \begin{cases} 2^\ell n - 2^\ell + 1 & \text{if } n \text{ is odd,} \\ 2^\ell n & \text{if } n \text{ is even,} \end{cases}$ and we want to prove that $f^{\ell+1}(n) = \begin{cases} 2^{\ell+1} - 2^{\ell+1}n & \text{if } n \text{ is odd,} \\ 1 - 2^{\ell+1}n & \text{if } n \text{ is even.} \end{cases}$

We get

$$\begin{aligned} f^{\ell+1}(n) &= f(f^\ell(n)) = \begin{cases} f(2^\ell n - 2^\ell + 1) & \text{if } n \text{ is odd,} \\ f(2^\ell n) & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} 2 - 2(2^\ell n - 2^\ell + 1) & \text{if } n \text{ is odd (since } 2^\ell n - 2^\ell + 1 \text{ is odd),} \\ 1 - 2(2^\ell n) & \text{if } n \text{ is even (since } 2^\ell n \text{ is even).} \end{cases} \\ &= \begin{cases} 2^{\ell+1} - 2^{\ell+1}n & \text{if } n \text{ is odd,} \\ 1 - 2^{\ell+1}n & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

so the inductive step works in this case.

Case (ii): ℓ is odd. This time we assume that $f^\ell(n) = \begin{cases} 2^\ell - 2^\ell n & \text{if } n \text{ is odd,} \\ 1 - 2^\ell n & \text{if } n \text{ is even,} \end{cases}$ and we want to prove that $f^{\ell+1}(n) = \begin{cases} 2^{\ell+1}n - 2^{\ell+1} + 1 & \text{if } n \text{ is odd,} \\ 2^{\ell+1}n & \text{if } n \text{ is even.} \end{cases}$

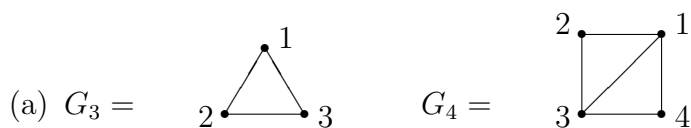
We get

$$\begin{aligned} f^{\ell+1}(n) &= f(f^\ell(n)) = \begin{cases} f(2^\ell - 2^\ell n) & \text{if } n \text{ is odd,} \\ f(1 - 2^\ell n) & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} 1 - 2(2^\ell - 2^\ell n) & \text{if } n \text{ is odd (since } 2^\ell - 2^\ell n \text{ is even),} \\ 2 - 2(1 - 2^\ell n) & \text{if } n \text{ is even (since } 1 - 2^\ell n \text{ is odd).} \end{cases} \\ &= \begin{cases} 2^{\ell+1}n - 2^{\ell+1} + 1 & \text{if } n \text{ is odd,} \\ 2^{\ell+1}n & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

so the inductive step works in this case too. Therefore the guessed formula is true for all integers $k \geq 1$.

2. For each integer $n \geq 3$, let G_n be the graph with vertex set $V(G_n) = \{1, 2, 3, \dots, n\}$ and where, for all distinct $a, b \in V(G_n)$, ab is an edge if and only if $\gcd(a, b) = 1$.

- Draw G_3 and G_4 .
- Find all integers $n \geq 3$ so that G_n has a Hamiltonian circuit.
- Show that G_n does not have an Euler circuit if $n \bmod 4 \neq 3$ ($n \not\equiv 3 \pmod{4}$). [*Hint:* do even n and odd n separately.]
- Suppose for each integer $n \geq 3$ we define the graph G'_n the same way as for G_n except that $V(G'_n) = \{2, 3, \dots, n\}$. Show (without a computer) that G'_8 does *not* have a Hamiltonian circuit. [*Hint:* start by thinking how 6 could fit into a Hamiltonian circuit.]

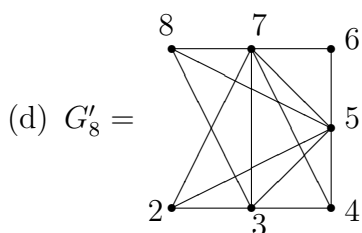


(b) For any integer $n \geq 3$, G_n has the Hamiltonian circuit $(1, 2, 3, \dots, n, 1)$, because $\gcd(k, k+1) = 1$ for any integer k , and also $\gcd(n, 1) = 1$.

(c) Notice that in the graph G_n , vertex 1 is connected to every other vertex, because $\gcd(1, k) = 1$ for every positive integer k . Thus vertex 1 has degree $n - 1$. So if n is even, vertex 1 has odd degree, and therefore G_n does not have an Euler circuit.

Now suppose n is odd. This means that $n \bmod 4 = 1$ or 3 . Suppose that $n \bmod 4 = 1$. This means that $n = 4q + 1$ for some integer q . Look at vertex 2. It is connected to exactly the odd vertices of G_n , because $\gcd(2, k) = 1$ exactly if k is odd. The odd vertices in G_n are $1, 3, 5, \dots, n = 4q + 1$, and there are $2q + 1$ of these. Thus vertex 2 has degree $2q + 1$, which is an odd number. Therefore G_n has no Euler circuit in the case $n \bmod 4 = 1$ either.

Note: Does G_n have an Euler circuit when $n \bmod 4 = 3$? Of course it does when $n = 3$, but it doesn't when $n = 7$. If you figure out whether G_n has an Euler circuit in any other cases ($n = 11, 15, 19, \dots$?) let your professor or TA know.



Suppose that G'_8 had a Hamiltonian circuit. Every vertex of G'_8 must be included in the circuit. Since the degree of vertex 6 is only 2, both of its edges must be in the circuit. So the circuit must contain vertices 5, 6, 7 in this order. What comes after 7? There are two cases.

Case (i): Suppose 8 comes after 7 in the circuit. Then the circuit contains 5, 6, 7, 8 in this order. The next vertex must then be 3, because 3 is the only vertex adjacent to 8 that is not in the circuit yet. So we have 5, 6, 7, 8, 3 in this order, and so the remaining vertices 2 and 4 must go after this and so must be next to each other, which is impossible because they are not adjacent. So no Hamiltonian circuit is possible in this case.

Case (ii): So suppose 8 does not come after 7 in the circuit. This means that 8 must come somewhere else in the circuit, but since we cannot use the edge 78 in the circuit we must use edges 85 and 83, because these are the only other edges containing vertex 8. So the circuit must contain 38567 in this order. But now once again vertices 2 and 4 have to come next, and so must be next to each other, which is impossible. So no Hamiltonian circuit is possible in this case either.

Therefore G'_8 does not have a Hamiltonian circuit.

Note: You can easily show (as in part (b)) that G'_n will have a Hamiltonian circuit whenever n is odd. Does G'_n have a Hamiltonian circuit for any **even** n ? If you get any answers, tell your professor or TA.

3. (a) Find the **total** number of all walks (starting at any vertex, ending at any vertex) of length 271 in the complete graph K_n .
- (b) Find the total number of walks of length 271 in the complete bipartite graph $K_{m,n}$.
- (c) Find the total number of *simple* paths of length $n - 1$ in K_n .
- (d) Find the total number of simple paths of length $m + n - 1$ in $K_{m,n}$. [You may assume that $m \geq n$. You will need to consider a few cases.]
- (a) In K_n we can start at any of the n vertices, then we have $n - 1$ choices for the first step of the walk (to any of the $n - 1$ vertices other than the one we are on), then $n - 1$ choices for the next step, and so on until we have taken 271 steps. So by the multiplication rule there are $n(n - 1)^{271}$ such walks.
- (b) Let the vertices of $K_{m,n}$ be grouped into the two sets M and N , where M has m vertices and N has n vertices. Then all steps must go between M and N . There are two cases.
- Case (i).* If the walk starts at a vertex of M , we have m choices for the starting vertex, then n choices for the first step of the walk (any of the vertices of N), then m choices for the next step of the walk (any of the vertices of M), and so on, alternating between n and m choices until we have taken 271 steps, with the 271st step having n choices. This means that (including the original choice of starting vertex) we will have n choices 136 times and m choices 136 times, so the total number of walks that start at a vertex of M is $n^{136}m^{136} = (nm)^{136}$.
- Case (ii).* If the walk starts at a vertex of N , exactly the same argument will again produce exactly $(nm)^{136}$ walks.
- So the total number of walks is $2(nm)^{136}$.
- (c) A simple path of length $n - 1$ in K_n will have to use each vertex exactly once, and we can use the n vertices in any order we like because all vertices are connected. Thus the total number of such paths is just the number of permutations of the numbers 1 to n , which is $n!$.
- (d) Since $K_{m,n}$ has $m + n$ vertices, any simple path of length $m + n - 1$ will use every vertex exactly once. Again let the vertices of $K_{m,n}$ be grouped into the two sets M and N , where M has m vertices and N has n vertices. Then all steps must go between M and N . Thus the only way to use up every vertex of $K_{m,n}$ with a simple path is if the sizes of M and N are at most one apart. In other words, if $m - n > 1$ then there are **no** such simple paths. We have two cases left.
- Case (i).* Suppose $m = n + 1$. Then all such simple paths must start and end at a vertex of the bigger set M . There are m choices for the starting vertex, then n choices for the first step, then $m - 1$ choices for the next step (any vertex of M except for the starting vertex), then $n - 1$ choices for the next step (any vertex of N except for the second vertex), and so on until we run out of vertices. Thus there are $m \cdot n \cdot (m - 1) \cdot (n - 1) \cdots = m!n! = n!(n + 1)!$ such simple paths in this case.
- Case (ii).* Suppose $m = n$. Then a simple path can start at any vertex. Suppose it starts at a vertex of M (m choices). Then the next step can be any of n choices in N , the next step any of the $m - 1$ remaining vertices in M , the next step any of

the remaining $n - 1$ vertices in N , and so on until we run out of vertices. This gives $m \cdot n \cdot (m - 1) \cdot (n - 1) \cdots = m!n! = (n!)^2$ such paths starting at a vertex of M . But we have the same number of paths starting at a vertex of N . So there are $2(n!)^2$ simple paths altogether in this case.