MATH 271 ASSIGNMENT 2 SOLUTIONS

1. (a) Prove **by induction** (or by well-ordering) that $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.

   (b) Prove **by induction** (or by well-ordering) that $(5/4)^n - (3/4)^n \ge n/2$ for all integers $n \ge 1$. [*Note*: you might have to consider the cases $n = 1, 2, 3$ and $n \ge 4$ separately.]

   (c) Prove that, for all real numbers $x \ge 2$, if $(5/4)^x - (3/4)^x \ge x/2$ then $3^x + 4^x \le 5^x$. Use this and part (b) to give another proof that $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.

(a) *Basis step.* When $n = 2$ the statement to be proved is $3^2 + 4^2 \le 5^2$, which is true since $9 + 16 = 25$.

*Inductive step.* Assume that $3^k + 4^k \le 5^k$ holds for some integer $k \ge 2$. We want to prove that $3^{k+1} + 4^{k+1} \le 5^{k+1}$. Well, we get

$$\begin{aligned} 5^{k+1} = 5 \cdot 5^k &\ge 5(3^k + 4^k) \quad \text{from the assumption} \\ &= 5 \cdot 3^k + 5 \cdot 4^k \\ &> 3 \cdot 3^k + 4 \cdot 4^k = 3^{k+1} + 4^{k+1} \end{aligned}$$

so the inductive step is proved.

Therefore $3^n + 4^n \le 5^n$ for all integers $n \ge 2$.

(b) *Basis step.* When $n = 1$ the statement to be proved is $\frac{5}{4} - \frac{3}{4} \ge \frac{1}{2}$ or $\frac{1}{2} \ge \frac{1}{2}$, which is true.

*Inductive step.* Assume that $(5/4)^k - (3/4)^k \ge k/2$ for some integer $k \ge 1$. We want to prove that $(5/4)^{k+1} - (3/4)^{k+1} \ge (k+1)/2$. From our assumption we get that $(5/4)^k \ge (3/4)^k + k/2$, so by multiplying both sides by $5/4$ we get

$$\left(\frac{5}{4}\right)^{k+1} = \frac{5}{4}\left(\frac{5}{4}\right)^k \ge \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5}{4} \cdot \frac{k}{2} = \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5k}{8}\ .$$

Thus

$$\left(\frac{5}{4}\right)^{k+1} - \left(\frac{3}{4}\right)^{k+1} \ge \frac{5}{4}\left(\frac{3}{4}\right)^k + \frac{5k}{8} - \left(\frac{3}{4}\right)^{k+1} = \left(\frac{5}{4} - \frac{3}{4}\right)\left(\frac{3}{4}\right)^k + \frac{5k}{8} = \frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8}\ ,$$

Now, *if we knew that*

$$\frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8} \ge \frac{k+1}{2}\ , \qquad (1)$$

then we would know that $\left(\frac{5}{4}\right)^{k+1} - \left(\frac{3}{4}\right)^{k+1} \ge \frac{k+1}{2}$, which is what we want to prove. So we need only prove (1) for all integers $k \ge 1$. Note that $\frac{1}{2}\left(\frac{3}{4}\right)^k > 0$, so $\frac{1}{2}\left(\frac{3}{4}\right)^k + \frac{5k}{8} > \frac{5k}{8}$, which means that to prove (1) we need only prove that $\frac{5k}{8} \ge \frac{k+1}{2}$. This is equivalent to $10k \ge 8k + 8$, or $2k \ge 8$, or $k \ge 4$. So we have proved (1) for all integers $k \ge 4$, which means we still have to prove (1) when $k = 1, 2$ and 3. We do this individually:

- When $k = 1$, (1) says $\frac{1}{2}\left(\frac{3}{4}\right) + \frac{5}{8} \ge 1$, which is true since $\frac{3}{8} + \frac{5}{8} = 1$.
- When $k = 2$, (1) says $\frac{1}{2}\left(\frac{9}{16}\right) + \frac{5}{4} \ge \frac{3}{2}$, which is true since $\frac{9}{32} + \frac{5}{4} = \frac{49}{32} > \frac{3}{2}$.
- When $k = 3$, (1) says $\frac{1}{2}\left(\frac{27}{64}\right) + \frac{15}{8} \ge 2$, which is true since $\frac{27}{128} + \frac{15}{8} = \frac{267}{128} > 2$.

This finishes the inductive step.

Since both the basis step and inductive step are now proved, we have proved that $(5/4)^n - (3/4)^n \geq n/2$ for all integers $n \geq 1$.

*Note.* Alternatively, we could have put $n = 1, 2, 3$ and 4 all into the basis step, then in the inductive step we would only need to consider the case $k \geq 4$. Or we could have handled the cases $n = 1, 2$ and 3 separately at the beginning, then use induction to prove the inequality for all integers $n \geq 4$ only, with only the case $n = 4$ in the basis step.

(c) Let $x$ be an arbitrary real number with $x \geq 2$, and assume that $(5/4)^x - (3/4)^x \geq x/2$. We want to prove that $3^x + 4^x \leq 5^x$. Well, from $(5/4)^x - (3/4)^x \geq x/2$ we multiply both sides by $4^x$ to get $5^x - 3^x \geq (\frac{x}{2})4^x$, then rearrange to get $5^x \geq 3^x + (\frac{x}{2})4^x$. Since $x \geq 2$, $x/2 \geq 1$, so $5^x \geq 3^x + (\frac{x}{2})4^x \geq 3^x + 4^x$, so $3^x + 4^x \leq 5^x$ as required. Done.

Now if $n \geq 2$ is an integer, then $n \geq 1$, so from part (b) we know that $(5/4)^n - (3/4)^n \geq n/2$. Therefore, since $n \geq 2$, we know from the first part of (c) that $3^n + 4^n \leq 5^n$.

2. The sequence $b_1, b_2, \ldots$ is defined by: $b_1 = 1$, and $b_n = \left\lceil \frac{n}{b_{n-1}} \right\rceil$ for all integers $n \geq 2$.

(a) Find $b_2, b_3, b_4, b_5$ and $b_6$.

(b) Use part (a) (and more data if you need it) to guess a simple formula for $b_n$ in terms of $n$. [*Hint*: do the cases of odd $n$ and even $n$ separately.]

(c) Use **induction** (or well-ordering) to prove your guess.

(d) Suppose the sequence $c_1, c_2, \ldots$ is defined by: $c_1 = 1, c_2 = 1$, and $c_n = \left\lceil \frac{n}{c_{n-2}} \right\rceil$ for all integers $n \geq 3$. Calculate enough terms of the sequence to enable you to see a pattern. Use that pattern to guess what $c_{271}$ and $c_{281}$ are. (No proof needed — yet.)

(a) We get

- $b_1 = 1$, $\quad b_2 = \lceil 2/b_1 \rceil = \lceil 2/1 \rceil = 2$,
- $b_3 = \lceil 3/b_2 \rceil = \lceil 3/2 \rceil = 2$, $\quad b_4 = \lceil 4/b_3 \rceil = \lceil 4/2 \rceil = 2$,
- $b_5 = \lceil 5/b_4 \rceil = \lceil 5/2 \rceil = 3$, $\quad b_6 = \lceil 6/b_5 \rceil = \lceil 6/3 \rceil = 2$.

(b) We could guess (maybe by further calculating that $b_7 = \lceil 7/b_6 \rceil = \lceil 7/2 \rceil = 4$ and $b_8 = \lceil 8/b_7 \rceil = \lceil 8/4 \rceil = 2$ for instance) that, for all integers $n \geq 1$,

$$b_n = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(c) *Basis step.* The formula for $b_n$ is correct when $n = 1$ (which is odd), because $(1+1)/2 = 1 = b_1$. This will turn out to be all we need for the basis step.

*Inductive step.* Assume that the formula for $b_n$ is true when $n = k$, where $k \geq 1$ is some integer. There are two cases:

- If $k$ is even, then we are assuming that $b_k = 2$, so

$$b_{k+1} = \left\lceil \frac{k+1}{b_k} \right\rceil = \left\lceil \frac{k+1}{2} \right\rceil = \frac{k+2}{2}$$

since $k + 1$ is odd, which agrees with the formula for $b_{k+1}$ when $n = k + 1$.

- If $k$ is odd, then we are assuming that $b_k = (k+1)/2$, so

$$b_{k+1} = \left\lceil \frac{k+1}{b_k} \right\rceil = \left\lceil \frac{k+1}{(k+1)/2} \right\rceil = \lceil 2 \rceil = 2,$$

  which agrees with the formula for $b_{k+1}$ when $n = k+1$ since $k+1$ is even.

So the formula is correct when $n = k+1$ in either case. This proves the inductive step.

Therefore by induction the formula for $b_n$ is correct for all integers $n \geq 1$.

(d) This time we get

- $c_3 = \lceil 3/1 \rceil = 3$, $c_4 = \lceil 4/1 \rceil = 4$, $c_5 = \lceil 5/3 \rceil = 2$, $c_6 = \lceil 6/4 \rceil = 2$,
- $c_7 = \lceil 7/2 \rceil = 4$, $c_8 = \lceil 8/2 \rceil = 4$, $c_9 = \lceil 9/4 \rceil = 3$, $c_{10} = \lceil 10/4 \rceil = 3$,
- $c_{11} = \lceil 11/3 \rceil = 4$, $c_{12} = \lceil 12/3 \rceil = 4$, $c_{13} = \lceil 13/4 \rceil = 4$, $c_{14} = \lceil 14/4 \rceil = 4$,
- $c_{15} = \lceil 15/4 \rceil = 4$, $c_{16} = \lceil 16/4 \rceil = 4$, $c_{17} = \lceil 17/4 \rceil = 5$, $c_{18} = \lceil 18/4 \rceil = 5$,
- $c_{19} = \lceil 19/5 \rceil = 4$, $c_{20} = \lceil 20/5 \rceil = 4$, $c_{21} = \lceil 21/4 \rceil = 6$, $c_{22} = \lceil 22/4 \rceil = 6$.

From this we guess that $c_n = 4$ whenever $n > 3$ is of the form $4k$ or $4k+3$ for some integer $k$, and $c_n = k+1$ whenever $n$ is of the form $4k+1$ or $4k+2$ for some integer $k$. For example, $21 = 4 \cdot 5 + 1$, so $c_{21} = 5 + 1 = 6$. From this pattern we would guess that since $271 = 4 \cdot 67 + 3$, $c_{271}$ should be $\mathbf{4}$, while since $281 = 4 \cdot 70 + 1$, $c_{281}$ should be $70 + 1 = \mathbf{71}$.

3. You are given the following "while" loop:

[*Pre-condition*: $m$ is a nonnegative integer, $a = 0$, $b = 0$, $i = 0$.]

**while** $(i \neq m)$

1. $b := a + b + 1$
2. $a := a - 4b$
3. $i := i + 1$

**end while**

[*Post-condition*: $b = m(-1)^{m+1}$.]

Loop invariant: $I(n)$ is

$$i = n, \quad a = \left\{ \begin{array}{ll} -2(n+1) & \text{if } n \text{ is odd} \\ 2n & \text{if } n \text{ is even} \end{array} \right\}, \quad b = n(-1)^{n+1}.$$

(a) Prove the correctness of this loop with respect to the pre- and post-conditions.

(b) Suppose the "while" loop is as above, except that statement 2 is replaced by: $a := a - b$. Run through the loop often enough, recording the various values of $a$ and $b$ that result, until you can predict what the post-condition value of $b$ will be when $m = 271$. What is your prediction? Explain.

(a) We first need to check that the loop invariant holds when $n = 0$. Since 0 is even, $I(0)$ says $i = 0$, $a = 2 \cdot 0 = 0$, and $b = 0(-1)^1 = 0$, and these are all true by the pre-conditions.

So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$ where $k < m$. We want to prove that $I(k+1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that

$$\left\{ \begin{array}{llll} i = k, & a = -2(k+1), & b = k(-1)^{k+1} = k & \text{if } k \text{ is odd,} \\ i = k, & a = 2k, & b = k(-1)^{k+1} = -k & \text{if } k \text{ is even,} \end{array} \right\}$$

and we now go through the loop.

- Step 1: $b := a + b + 1 = \left\{ \begin{array}{ll} -2(k+1) + k + 1 & \text{if } k \text{ is odd} \\ 2k - k + 1 & \text{if } k \text{ is even} \end{array} \right\}$

  $$= \left\{ \begin{array}{ll} (k+1)(-1) & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{array} \right\} = (k+1)(-1)^{k+2},$$

  which agrees with the formula for $b$ in $I(k+1)$.
- Step 2: $a := a - 4b = \left\{ \begin{array}{ll} -2(k+1) - 4(-k-1) & \text{if } k \text{ is odd} \\ 2k - 4(k+1) & \text{if } k \text{ is even} \end{array} \right\}$

  $$= \left\{ \begin{array}{ll} 2(k+1) & \text{if } k+1 \text{ is even} \\ -2(k+2) & \text{if } k+1 \text{ is odd} \end{array} \right\},$$

  which agrees with the formula for $a$ in $I(k+1)$.
- Step 3: $i := i + 1 = k + 1$, which agrees with $I(k+1)$.

Thus $I(k+1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $b = m(-1)^{m+1}$ as required.

(b) If we set the variables to their pre-condition values of $a = 0$, $b = 0$ and $i = 0$, and run through the loop, the new values we get are $b = 0 + 0 + 1 = 1$, $a = 0 - 1 = -1$, $i = 1$. If we continue to run through the loop, and keep track of the variables in a table, here is what we get:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $b$ | 0 | 1 | 1 | 0 | -1 | -1 | 0 |
| $a$ | 0 | -1 | -2 | -2 | -1 | 0 | 0 |
| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

At this point (when $n = 6$) our values of $b$ and $a$ are back to what they were at the beginning (when $n = 0$), namely $b = a = 0$. Since the loop calculates the new values of $a$ and $b$ only in terms of their old values, and not in terms of $n$ for example, the values of $a$ and $b$ should continue to cycle through the same values in the above table. This means that $a = b = 0$ whenever $n$ is a multiple of 6, $a = -1$ and $b = 1$ whenever $n$ is 1 plus a multiple of 6, and so on. Since $271 = 6 \cdot 45 + 1$, when the loop ends (at $m = i = n = 271$), we should have $b = \mathbf{1}$.