MATH 271 ASSIGNMENT 4 SOLUTIONS

1. If $f : X \to X$ is a function, define $f^2(x)$ to be $(f \circ f)(x)$, and inductively define $f^k(x) = (f \circ f^{k-1})(x)$ for each integer $k \geq 3$. (So $f^3(x) = (f \circ f^2)(x) = f(f(f(x)))$ for instance.) We also define $f^1(x)$ to be $f(x)$.

   Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by: for all $n \in \mathbb{Z}$, $f(n) = n + 1 + 2(-1)^n = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$

   (a) Find $f^2(n)$, $f^3(n)$, and $f^4(n)$.

   (b) Use part (a) (and more data if you need it) to guess a fairly simple formula for $f^k(n)$ for any positive integer k. (You may need to consider k odd and k even separately.)

   (c) Use induction on k (or well ordering) to prove your guess.

   (d) Use your formula for $f^k(n)$ to find $f^{2008}(271)$.

   (e) Define $g : \mathbb{Z} \to \mathbb{Z}$ by: for all $n \in \mathbb{Z}$, $g(n) = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ 1-n & \text{if } n \text{ is odd.} \end{cases}$

   Calculate $g^2(n)$, $g^3(n)$, and $g^4(n)$, and use them (and more data if you need it) to predict what $g^{2008}(271)$ is.

   (a) We get

$$\begin{aligned} f^2(n) &= f(f(n)) = f(n + 1 + 2(-1)^n) \\ &= (n + 1 + 2(-1)^n) + 1 + 2(-1)^{n+1+2(-1)^n} \\ &= n + 2 + 2(-1)^n + 2(-1)^{n+1} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n + 2 \qquad \text{since } n \text{ and } n+1 \text{ are of opposite parity,} \end{aligned}$$

$$f^3(n) = f(f^2(n)) = f(n+2) = (n+2) + 1 + 2(-1)^{n+2} = n + 3 + 2(-1)^n,$$

$$\begin{aligned} f^4(n) &= f(f^3(n)) = f(n + 3 + 2(-1)^n) \\ &= (n + 3 + 2(-1)^n) + 1 + 2(-1)^{n+3+2(-1)^n} \\ &= n + 4 + 2(-1)^n + 2(-1)^{n+3} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n + 4 \qquad \text{since } n \text{ and } n+3 \text{ are of opposite parity.} \end{aligned}$$

   (b) From part (a) we would guess that

$$f^k(n) = \begin{cases} n + k + 2(-1)^n & \text{if } k \text{ is odd,} \\ n + k & \text{if } k \text{ is even.} \end{cases}$$

   (c) *Basis step.* Our guessed formulas for $f^k(n)$ are true for $k = 1, 2, 3$ and 4, by part (a).

   Inductive step. Assume that our guessed formula is true for some integer $k = \ell \geq 1$. We want to prove that our formula is true when $k = \ell + 1$. We do this in two cases:

   Case (i): ℓ is even. So we assume that $f^\ell(n) = n + \ell$, and we want to prove that $f^{\ell+1}(n) = n + \ell + 1 + 2(-1)^n$.

Well, we get

$$f^{\ell+1}(n) = f(f^\ell(n)) = f(n+\ell) = n+\ell+1+2(-1)^{n+\ell} = n+\ell+1+2(-1)^n,$$

since ℓ is even, so the inductive step works in this case.
Case (ii): ℓ is odd. This time we assume that $f^\ell(n) = n+\ell+2(-1)^n$, and we want to prove that $f^{\ell+1}(n) = n+\ell+1$. We get

$$\begin{aligned} f^{\ell+1}(n) &= f(f^\ell(n)) = f(n+\ell+2(-1)^n) \\ &= (n+\ell+2(-1)^n)+1+2(-1)^{n+\ell+2(-1)^n} \\ &= n+\ell+1+2(-1)^n+2(-1)^{n+\ell} \qquad \text{since } 2(-1)^n \text{ is even} \\ &= n+\ell+1 \qquad \text{since } n \text{ and } n+\ell \text{ are of opposite parity,} \end{aligned}$$

so the inductive step works in this case too. Therefore the guessed formula is true for all integers $k \geq 1$.

(d) By the formula, since 2008 is even, $f^{2008}(271) = 271 + 2008 = 2279$.

(e) We get

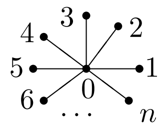
$$\begin{aligned} g^2(n) &= g(g(n)) = \begin{cases} g(n+3) & \text{if } n \text{ is even,} \\ g(1-n) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} 1-(n+3) & \text{if } n \text{ is even (since } n+3 \text{ is odd),} \\ (1-n)+3 & \text{if } n \text{ is odd (since } 1-n \text{ is even).} \end{cases} \\ &= \begin{cases} -n-2 & \text{if } n \text{ is even,} \\ -n+4 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$\begin{aligned} g^3(n) &= g(g^2(n)) = \begin{cases} g(-n-2) & \text{if } n \text{ is even,} \\ g(-n+4) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} (-n-2)+3 & \text{if } n \text{ is even (since } -n-2 \text{ is even),} \\ 1-(-n+4) & \text{if } n \text{ is odd (since } -n+4 \text{ is odd).} \end{cases} \\ &= \begin{cases} -n+1 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

$$\begin{aligned} g^4(n) &= g(g^3(n)) = \begin{cases} g(-n+1) & \text{if } n \text{ is even,} \\ g(n-3) & \text{if } n \text{ is odd.} \end{cases} \\ &= \begin{cases} 1-(-n+1) & \text{if } n \text{ is even (since } -n+1 \text{ is odd),} \\ (n-3)+3 & \text{if } n \text{ is odd (since } n-3 \text{ is even).} \end{cases} \\ &= n \qquad \text{for all } n \in \mathbb{Z}. \end{aligned}$$

Now $g^5(n) = g(g^4(n)) = g(n)$, and so $g^6(n) = g(g^5(n)) = g(g(n)) = g^2(n)$, and so on; the formulas for $g^k(n)$ will cycle through the above four functions forever. In particular, since 2008 is a multiple of 4, $g^{2008}(n)$ will equal n for all n, so $g^{2008}(271) = 271$.

2. For each integer $n \geq 2$, let S_n be the "star-like" graph shown at the right, where there are $n+1$ vertices altogether, including the one in the middle.



(a) Find a formula (in terms of n) for the number of paths of length 2 in S_n. [*Hint*: how many such paths start at each vertex?]

(b) Find a formula (in terms of n) for the number of walks of length 2 in S_n.

(c) Write out the $(n+1) \times (n+1)$ adjacency matrix M_n of S_n in general. Then find M_n^2, and explain what it has to do with your answer to part (b).

(d) Prove that for any simple graph G, the number of walks in G of length 2 is always even. [*Hint*: how can you pair up the walks?]

(a) Since paths cannot repeat vertices, there are no paths of length 2 starting at vertex 0, because once we go from 0 to one of the other vertices we are stuck. If we start at one of the other vertices, say at vertex 1, then we must go to vertex 0, and from there we have $n-1$ choices for the third vertex, namely any vertex except vertex 0 or 1. So there are $n-1$ paths of length 2 starting from vertex 1. By symmetry there are $n-1$ paths of length 2 starting from any of the vertices 1 to n, so there are $n(n-1)$ paths of length 2 in S_n altogether.

(b) For walks we are allowed to repeat vertices, so if we start at vertex 0 we can go to any of the vertices 1 to n, and then we must go back to 0. So there are n walks of length 2 starting at vertex 0. If we start at vertex 1 instead, then we must go to vertex 0, and then we can go to any of the vertices 1 to n, so there are n walks of length 2 starting at vertex 1. Again by symmetry there are n walks of length 2 starting from any of the vertices 1 to n, so there are $n + n(n) = n + n^2$ walks of length 2 in S_n.

(c) If we order the rows and columns in the natural way (with the vertices in the order $0, 1, 2, \ldots, n$), we will get

$$M_n = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \text{and so} \qquad M_n^2 = \begin{bmatrix} n & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

The matrix M_n^2 tells us how many walks of length 2 the graph S_n has between all possible pairs of vertices. Since the sum of all the entries in M_n^2 is $n + n^2$, this must be the number of walks of length 2 in S_n, which agrees with our answer in part (b).

(d) In a simple graph G, every walk of length 2 is either of the form abc, where a, b, c are three different vertices, or of the form aba where a and b are different vertices. Walks of the first kind can be paired up by pairing each walk abc with the walk cba. Walks of the second kind can be paired up by pairing each walk aba with the walk bab. Thus all walks of length 2 are paired up, so there must be an even number of them altogether.

3. For each integer $n \geq 3$, let G_n be the graph whose vertices are all two-element subsets of $\{1, 2, \ldots, n\}$, and with edges defined as follows: for any vertices A and B of G_n (so A and B are two-element subsets of $\{1, 2, \ldots, n\}$), A and B are adjacent if and only if $N(A \cap B) = 1$ (where $N(X)$ is the number of elements in the set X).

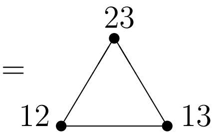
   (a) Draw the graphs G_3 and G_4. [You can label the vertex $\{i, j\}$ as just ij if you like.]

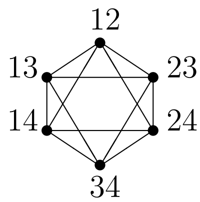
   (b) For each integer $n \geq 3$, find and prove formulas (in terms of n) for the number of vertices in G_n, the degree of each vertex, and the number of edges in G_n.

   (c) For which n does G_n have an Euler circuit? Explain.

   (d) For which n does G_n have a Hamiltonian circuit? Explain. [*Hint*: induction on n.]

(a) For G_3 the vertices are all two-element subsets of $\{1, 2, 3\}$, so they are 12, 13 and 23 (writing the subset $\{1, 2\}$ as just 12 for instance). For G_4 the vertices are similarly 12, 13, 14, 23, 24 and 34. We get

$G_3 =$ 

$G_4 =$ 

(b) The number of vertices in G_n is $\binom{n}{2}$, the number of 2-element subsets of $\{1, 2, \ldots, n\}$.

The degree of the vertex 12 of G_n is the number of 2-element subsets of $\{1, 2, \ldots, n\}$ which contain either 1 or 2 (but not both). There are $n - 2$ 2-element subsets that contain 1 but not 2, and $n - 2$ 2-element subsets that contain 2 but not 1, so the degree of 12 must be $2(n - 2) = 2n - 4$. By symmetry the degree of every vertex of G_n is $2n - 4$.

From above, the sum of the degrees of the vertices of G_n must be the number of vertices times the degree of each vertex, which is $\binom{n}{2}(2n - 4)$. This is twice the number of edges, so the number of edges in G_n must be

$$\frac{1}{2}\binom{n}{2}(2n-4) = \binom{n}{2}(n-2) = \frac{n(n-1)(n-2)}{2} .$$

(c) It is clear that G_n is connected, because if ab and cd are arbitrary nonadjacent vertices of G_n, then ab, ad, cd is a walk in G_n from ab to cd.

Now, since the degree of each vertex of G_n is $2n - 4$ which is even, G_n will have an Euler circuit for all integers $n \geq 3$.

(d) We prove by induction on n that G_n has a Hamiltonian circuit for each integer $n \geq 3$.

Basis step. It is easy to see from the graph that G_3 has a Hamiltonian circuit, for example $12, 13, 23, 12$ is a Hamiltonian circuit in G_3.

Inductive step. Assume that G_k has a Hamiltonian circuit for some integer $k \geq 3$. This means that there is a circuit in G_k containing each vertex exactly once (except that the first vertex equals the last vertex). We can start this circuit at any vertex we like, so let's say we start it at the vertex 12. The second vertex in the circuit must be a 2-element subset of $\{1, 2, \ldots, k\}$ which contains either 1 or 2 (but not both), so by symmetry we can assume it is $1k$. So the Hamiltonian circuit starts off $12, 1k$ and so on, eventually ending back at 12 after going through each vertex of G_k exactly once.

We want to find a Hamiltonian circuit in G_{k+1}, and we will use the Hamiltonian circuit in G_k to do this. The vertices of G_{k+1} which are not vertices of G_k are just the 2-element subsets of $\{1, 2, \ldots, k+1\}$ which contain $k+1$, so they are $1(k+1), 2(k+1), \ldots, k(k+1)$. All the edges in G_k are still in G_{k+1}, so we replace the edge $12, 1k$ of the Hamiltonian circuit in G_k by the path $12, 1(k + 1), 2(k + 1), \ldots, k(k + 1), 1k$, and then we will get a Hamiltonian circuit in G_{k+1}. Note that this is allowed, since each two consecutive vertices of the path $12, 1(k + 1), 2(k + 1), \ldots, k(k + 1), 1k$ have an element in common, so they are adjacent in G_{k+1}.

For example, the Hamiltonian circuit $12, 13, 23, 12$ in G_3 would be used to make the Hamiltonian circuit $12, 14, 24, 34, 13, 23, 12$ in G_4, by replacing the edge $12, 13$ by the path $12, 14, 24, 34, 13$ in G_4, which contains all the vertices in G_4 which are not in G_3.

This completes the inductive step. By induction, G_n has a Hamiltonian circuit for every integer $n \geq 3$.