MATH 271 ASSIGNMENT 4

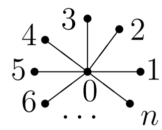
Due 4:00 PM Friday, April 11, 2008. Hand your assignment (stapled please) to the marker (Jason Nicholson) in his office MS 390. (Or if nobody is in his office, put your assignment under the door.) Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. If $f : X \to X$ is a function, define $f^2(x)$ to be $(f \circ f)(x)$, and inductively define $f^k(x) = (f \circ f^{k-1})(x)$ for each integer $k \geq 3$. (So $f^3(x) = (f \circ f^2)(x) = f(f(f(x)))$ for instance.) We also define $f^1(x)$ to be $f(x)$.

   Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by: for all $n \in \mathbb{Z}$, $f(n) = n+1+2(-1)^n = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$

   (a) Find $f^2(n)$, $f^3(n)$, and $f^4(n)$.
   (b) Use part (a) (and more data if you need it) to guess a fairly simple formula for $f^k(n)$ for any positive integer k. (You may need to consider k odd and k even separately.)
   (c) Use induction on k (or well ordering) to prove your guess.
   (d) Use your formula for $f^k(n)$ to find $f^{2008}(271)$.
   (e) Define $g : \mathbb{Z} \to \mathbb{Z}$ by: for all $n \in \mathbb{Z}$, $g(n) = \begin{cases} n+3 & \text{if } n \text{ is even,} \\ 1-n & \text{if } n \text{ is odd.} \end{cases}$
   Calculate $g^2(n)$, $g^3(n)$, and $g^4(n)$, and use them (and more data if you need it) to predict what $g^{2008}(271)$ is.



2. For each integer $n \geq 2$, let S_n be the "star-like" graph shown at the right, where there are $n+1$ vertices altogether, including the one in the middle.

   (a) Find a formula (in terms of n) for the number of paths of length 2 in S_n. [*Hint*: how many such paths start at each vertex?]
   (b) Find a formula (in terms of n) for the number of walks of length 2 in S_n.
   (c) Write out the $(n+1) \times (n+1)$ adjacency matrix M_n of S_n in general. Then find M_n^2, and explain what it has to do with your answer to part (b).
   (d) Prove that for any simple graph G, the number of walks in G of length 2 is always even. [*Hint*: how can you pair up the walks?]

3. For each integer $n \geq 3$, let G_n be the graph whose vertices are all two-element subsets of $\{1, 2, \ldots, n\}$, and with edges defined as follows: for any vertices A and B of G_n (so A and B are two-element subsets of $\{1, 2, \ldots, n\}$), A and B are adjacent if and only if $N(A \cap B) = 1$ (where $N(X)$ is the number of elements in the set X).

   (a) Draw the graphs G_3 and G_4. [You can label the vertex $\{i, j\}$ as just ij if you like.]
   (b) For each integer $n \geq 3$, find and prove formulas (in terms of n) for the number of vertices in G_n, the degree of each vertex, and the number of edges in G_n.
   (c) For which n does G_n have an Euler circuit? Explain.
   (d) For which n does G_n have a Hamiltonian circuit? Explain. [*Hint*: induction on n.]