THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271
FINAL EXAMINATION, WINTER 2006
TIME: 3 HOURS

NAME________________________________ID____________Section______

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Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(104, 81)$. Also use the algorithm to find integers x and y such that $\gcd(104, 81) = 104x + 81y$.

(b) Use part (a) to find an inverse for 81 modulo 104.

[12] 2. Let **Z** be the set of all integers, and let $\mathcal{S}$ be the statement

"For all subsets A and B of **Z**, if $5 \in A - B$ or $5 \in B - A$ then $5 \notin A \cap B$."

(a) Is $\mathcal{S}$ true? Give a proof or disproof.

(b) Write out the *contrapositive* of $\mathcal{S}$, and give a proof or disproof.

(c) Write out the *converse* of $\mathcal{S}$, and give a proof or disproof.

[11] 3. Let $\mathbf{N}$ be the set of all *positive* integers, and define the relation R on $\mathbf{N}$ by:

$$\text{for any } x, y \in \mathbf{N},\ xRy \text{ if and only if } \gcd(x, y) > 1.$$

(a) Is R reflexive? Symmetric? Transitive? Give reasons.

(b) Is R an equivalence relation? Explain.

(c) Prove or disprove: $\forall x \in \mathbf{N}\ \exists y \in \mathbf{N}$ so that xRy.

(d) Prove or disprove: $\forall x \in \mathbf{N}\ \exists y \in \mathbf{N}$ so that $x \not R\, y$ (that is, x is **not** related to y).

[9] 4. Let $A = \{1, 2, \ldots, 10\}$, and let $\mathcal{F}$ be the set of all functions $f : A \to A$. Define a relation R on $\mathcal{F}$ by:

for all $f, g \in \mathcal{F}$, fRg if and only if there exists some $i \in A$ so that $f(i) = g(i)$.

(a) Is R transitive? Explain.

(b) Let $g \in \mathcal{F}$ be defined by $g(i) = 1$ for all $i \in A$. Find the number of functions $f \in \mathcal{F}$ so that fRg.

(c) How many of the functions f in part (b) are one-to-one? Explain.

[5] 5. If A is a finite set, $N(A)$ denotes the number of elements in A. Let $X = \{1, 2, \ldots, 271\}$, and let $\mathcal{P}(X)$ denote the power set of X. One of the following two statements is true and one is false. Prove the true statement and disprove the false statement.

(a) $\exists A \in \mathcal{P}(X)$ so that $\forall B \in \mathcal{P}(X)$, $N(A \cup B)$ is even.

(b) $\exists A \in \mathcal{P}(X)$ so that $\forall B \in \mathcal{P}(X)$, $N(A \cup B)$ is odd.

[6] 6. Suppose that A and B are sets, and that $(1, 2)$ and $(2, 3)$ are elements of $A \times B$. Find the smallest possible number of elements in (a) $A \times B$; (b) $A \cap B$; (c) $A - B$.

[13] 7. Let $\mathcal{B}_{\leq 10}$ denote the set of all binary strings (sequences of 0's and 1's) of length at most 10. Define the relation R on $\mathcal{B}_{\leq 10}$ by:

for all $s, t \in \mathcal{B}_{\leq 10}$, sRt if and only if the number of 1's in s equals the number of 1's in t.

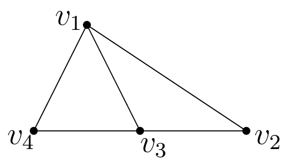
(a) Prove that R is an equivalence relation on $\mathcal{B}_{\leq 10}$.

(b) Find three elements of $\mathcal{B}_{\leq 10}$ belonging to the equivalence class [101].

(c) Find the number of strings of length 6 in [101]. Simplify your answer.

(d) Find the number of distinct equivalence classes of R.

[9] 8. Let G be the graph



(a) Find the adjacency matrix M of G.

(b) **Use the matrix** M to find the number of walks in G of length 2 between v_1 and v_3.

(c) Does G have a Eulerian circuit? Explain.

(d) Does G have a Hamiltonian circuit? Explain.

[7] 9. Prove by mathematical induction that $4 \mid (9^n - 1)$ for all integers $n \geq 0$.

$\overline{[80]}$