MATHEMATICS 271 MID-TERM TEST SOLUTIONS Wednesday, 9 March 2005

SHOW ALL WORK. Marks for each problem are to the left of the problem number.
NO CALCULATORS PLEASE.

[4] 1. **Use the Euclidean algorithm** to find $\gcd(74, 35)$.

Recall from lecture: if a and b are positive integers, then

$$\gcd(a, b) = \gcd(b, a \textbf{ mod } b).$$

By repeated application of this principle, we have the following:

$$\begin{array}{rcrcl}
74 = 2 \cdot 35 + 4 & \Longrightarrow & 74 \textbf{ mod } 35 = 4 & \Longrightarrow & \gcd(74, 35) = \gcd(35, 4) \\
35 = 8 \cdot 4 + 3 & \Longrightarrow & 35 \textbf{ mod } 4 = 3 & \Longrightarrow & \gcd(35, 4) = \gcd(4, 3) \\
4 = 1 \cdot 3 + 1 & \Longrightarrow & 4 \textbf{ mod } 3 = 1 & \Longrightarrow & \gcd(4, 3) = \gcd(3, 1) \\
3 = 3 \cdot 1 + 0 & \Longrightarrow & 3 \textbf{ mod } 1 = 0 & \Longrightarrow & \gcd(3, 1) = \gcd(1, 0) = 1.
\end{array}$$

Therefore,

$$\gcd(74, 35) = 1.$$

[7] 2. One of the following statements is true and one is false. Prove the true statement **by contradiction**. Give a counterexample for the false statement.

(a) For all sets A and B, if $3 \notin A$ then $3 \notin A \cup B$.

This statement is false. Here is a counter-example: let $A = \emptyset$ and let $B = \{3\}$. Note that $A \cup B = \{3\}$. Then $3 \notin A$ and $3 \in A \cup B$.

(b) For all sets A and B, if $3 \notin A$ then $3 \notin A \cap B$.

This statement is true.

Proof: Suppose (for a contradiction) that the negation is true; in other words, suppose there exist sets A and B such that $3 \notin A$ and $3 \in A \cap B$. Since $3 \in A \cap B$ it follows from the definition of set intersection that $3 \in A$ and $3 \in B$. Since this contradicts $3 \notin A$, it follows that the negation is false. QED

Another (slightly different) proof starts off directly. Let A and B be sets so that $3 \notin A$. We want to prove that $3 \notin A \cap B$. Suppose (for a contradiction) that $3 \in A \cap B$. This means that $3 \in A$ and $3 \in B$. But $3 \in A$ contradicts $3 \notin A$. Thus $3 \notin A \cap B$. Done.

[11] 3. Let $\mathcal{S}$ be the statement: for all integers a, if $6 \mid a$ then $6 \mid (3a - 12)$.
(a) Prove directly from the definition of divisibility that $\mathcal{S}$ is true.

Let a be an arbitrary (but fixed) integer. Suppose $6 \mid a$. Then $a = 6k$ for some $k \in \mathbb{Z}$. Thus, $3a - 12 = 18k - 12 = 6(3k - 2)$. Since $3k - 2$ is an integer, it follows from $3a - 12 = 6(3k - 2)$ that $6 \mid (3a - 12)$. QED

(b) Write out the *converse* of statement $\mathcal{S}$, and give a proof or disproof.

The converse of statement $\mathcal{S}$ is the following: For all integers a, if $6 \mid (3a - 12)$ then $6 \mid a$.

Proof: This statement is false, and here is a counterexample. Let $a = 2$. Then $3a - 12 = 6 - 12 = -6$, and so $6 \mid (3a - 12)$. However $6 \nmid 2$, so $6 \nmid a$, so the converse of $\mathcal{S}$ is false.

(c) Write out the *contrapositive* of statement $\mathcal{S}$, and give a proof or disproof.

The contrapositive of statement $\mathcal{S}$ is the following: For all integers a, if $6 \nmid (3a - 12)$ then $6 \nmid a$. This is true since it is logically equivalent to statement $\mathcal{S}$, which is true (see (a) above).

[6] 4. In this problem, you may assume that every integer is either even or odd but not both.

(a) Prove or disprove the statement:

"For all integers a, either $a + 4$ is odd or $4a + 1$ is even."

The statement is false. To see this, prove the negation: "There is some integer a so that $a + 4$ is even **and** *$4a + 1$ is odd."*

Proof: Let $a = 0$. Then $a + 4 = 4$ is even and $4a + 1 = 1$ is odd. QED

(b) Write out the *negation* of the statement in (a). Is it true or false?

The negation of "For all integers a, either $a + 4$ is odd or $4a + 1$ is even" is "There is some integer a so that $a + 4$ is even and $4a + 1$ is odd". The negation is true, as shown in part (a) above.

[5] 5. Prove or disprove the following two statements:

(a) $\forall$ sets A $\exists$ a set B so that $A - B = \emptyset$.

The statement is true.

Proof: Let A be an arbitrary (but fixed) set. Define $B := A$. Then $A - B = A - A = \emptyset$. QED

(b) $\forall$ sets A $\exists$ a set B so that $A - \{1\} = B - \{2\}$.

The statement is false. To see this, prove the negation: "There is a set A such that for every set B, $A - \{1\} \neq B - \{2\}$.

Proof: Let $A = \{2\}$. Then $A - \{1\} = \{2\}$. Thus, $2 \in A - \{1\}$. For every set B, $2 \notin B - \{2\}$, by the definition of set difference. Thus, $2 \in A - \{1\}$ and $2 \notin B - \{2\}$, from which it follows immediately that $A - \{1\} \neq B - \{2\}$. QED

[7] 6. The sequence $a_1, a_2, a_3, \ldots$ is defined by: $a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} + 5a_{n-2}$ for all integers $n \geq 3$. Prove **using strong mathematical induction** that $a_n \geq 3^{n-1}$ for all integers $n \geq 3$.

Let $P(n)$ be the predicate: $a_n \geq 3^{n-1}$. We will prove:

$$\forall n \in \mathbb{Z}, \qquad n \geq 3 \textit{ implies } P(n).$$

I. Base Case:

(i) *Suppose $n = 3$. Then $a_n = a_3 = 2a_2 + 5a_1 = 2 \cdot 2 + 5 \cdot 1 = 9$. On the other hand $3^{n-1} = 3^{3-1} = 3^2 = 9$. Since $9 \geq 9$, it follows that $a_n \geq 3^{n-1}$ when $n = 3$. Thus, $P(3)$ is true.*

(ii) *Suppose $n = 4$. Then $a_n = a_4 = 2a_3 + 5a_2 = 2 \cdot 9 + 5 \cdot 2 = 28$. On the other hand $3^{n-1} = 3^{4-1} = 3^3 = 27$. Since $28 \geq 27$, it follows that $a_n \geq 3^{n-1}$ when $n = 4$. Thus, $P(4)$ is true.*

II. Inductive Step: *Let k be an integer and $k \geq 5$. Suppose, for all integers i, if $3 \leq i \leq k$ then $P(i)$ is true. (This is the* **inductive hypothesis**.*) We will show that $P(k+1)$ is true.*

$$\begin{aligned}
a_{k+1} &= 2a_k + 5a_{k-1}, && \textit{by definition above} \\
&\geq 2(3^{k-1}) + 5(3^{k-2}), && \textit{using } P(k) \textit{ and } P(k-1) \\
&= 2(3^{k-1}) + (3+2)(3^{k-2}) \\
&= 2 \cdot 3^{k-1} + 3 \cdot 3^{k-2} + 2 \cdot 3^{k-2} \\
&= 2 \cdot 3^{k-1} + 3^{k-1} + 2 \cdot 3^{k-2} \\
&= 3 \cdot 3^{k-1} + 2 \cdot 3^{k-2} \\
&= 3^k + 2 \cdot 3^{k-2}.
\end{aligned}$$

Since $3^{k-2} > 0$, therefore

$$3^k + 2 \cdot 3^{k-2} \quad > 3^k.$$

Therefore

$$a_{k+1} \geq 3^k;$$

in other words, $P(k+1)$ is true.

By the Principle of Strong Mathematical Induction, it follows that $a_n \geq 3^{n-1}$ for all integers $n \geq 3$.

[40]