MATH 271 ASSIGNMENT 1 SOLUTIONS

1.

(a) Let $\mathcal{S}$ be the statement

For all integers n, if n is even then $3n - 11$ is odd.

Is $\mathcal{S}$ true? Give a proof or counterexample.

(b) Write out the *contrapositive* of statement $\mathcal{S}$, and give a proof or disproof.

(c) Write out the *converse* of statement $\mathcal{S}$, and give a proof or disproof.

(d) Prove or disprove the statement

For all integers n, if n is odd then $2n - 11$ is even.

Then write out the converse of this statement and prove or disprove it.

(a) $\mathcal{S}$ is **true**. Here is a proof.

Let n be an arbitrary even integer. This means that $n = 2k$ for some integer k. Then

$$3n - 11 = 3(2k) - 11 = 6k - 11 = 2(3k - 6) + 1$$

where $3k - 6$ is an integer. Therefore $3n - 11$ is odd by the definition of odd.

(b) The contrapositive of $\mathcal{S}$ is:

For all integers n, if $3n - 11$ is not odd then n is not even,

which could also be written (using a result on page 159 of the text)

For all integers n, if $3n - 11$ is even then n is odd.

It is **true**, because it is equivalent to the original statement $\mathcal{S}$ which is true.

(c) The converse of $\mathcal{S}$ is

For all integers n, if $3n - 11$ is odd then n is even.

This statement is **true**. Here is a proof.

Assume that $3n - 11$ is odd, where n is an integer. This means that $3n - 11 = 2k + 1$ for some integer k. We can rewrite this equation as $n = 2k + 12 - 2n = 2(k + 6 - n)$, where $k + 6 - n$ is an integer since k and n are integers. Therefore n equals 2 times an integer, so n is even.

Note. The converse could also be proven by writing its contrapositive

For all integers n, if n is not even then $3n - 11$ is not odd

in the form

For all integers n, if n is odd then $3n - 11$ is even

and proving this.

(d) This statement is **false**. A counterexample is $n = 1$. For then n is odd, but $2n - 11 = 2 - 11 = -9$ is not even.

The converse of this statement is

For all integers n, if $2n - 11$ is even then n is odd.

This statement is **true** vacuously. For every integer n, $2n - 11 = 2(n - 6) + 1$ where $n - 6$ is an integer, thus $2n - 11$ is odd and so cannot be even. Since the "if" part of the conditional never holds, the statement is true vacuously.

2. Prove or disprove the following statements:

(a) There exists a prime number a such that $a + 271$ is prime.

(b) There exists a prime number a such that $a + 271$ is composite.

(c) There exists a composite number a such that $a + 271$ is prime.

(d) There exists a composite number a such that $a + 271$ is composite.

(e) Choose one of statements (a) to (d) (your choice), replace 271 with your U of C ID number, and prove or disprove the resulting statement.

(a) This statement is **false**. Here is a proof.

Assume a is a prime number. We have two cases.

Case (i): $a = 2$. Then $a + 271 = 2 + 271 = 273 = 3 \cdot 91$, so $a + 271$ is not prime.

Case (ii): $a > 2$. Then a must be odd, so $a = 2k + 1$ for some integer k. Then $a + 271 = 2k + 1 + 271 = 2k + 272 = 2(k + 136)$, where $k + 136$ is an integer. Therefore $a + 271$ is not prime.

In neither case can we get that $a + 271$ is prime, so the statement is false.

(b) This statement is **true**. An example is $a = 3$. Then a is prime and $a + 271 = 274 = 2 \cdot 137$ is composite.

(c) This statement is **true**. An example is $a = 6$. Then a is composite and $a + 271 = 277$ is prime (it turns out).

Note. An alternate proof would go like this: since there are infinitely many primes (Theorem 3.7.4 of the text), there must be a prime $p \geq 275$. Then p is odd, so $p - 271$ must be even (prove it), and $p - 271 \geq 4$, so $p - 271$ is composite. Put $a = p - 271$; then a is composite and $a + 271 = p$ is prime.

(d) This statement is **true**. An example is $a = 9$. Then a is composite and $a + 271 = 280$ is composite too.

(e) Regardless of what your ID number is, probably (d) is the easiest statement to prove. Let's do it for the hypothetical ID number 123456. Choosing $a = 4$, we get that a is composite and that $a + 123456 = 123460$ is also composite.

3. *Note*: $\mathbf{Z}$ denotes the set of all integers, and $\mathbf{Z}^+$ denotes the set of all positive integers.

(a) Prove the following statements:

(i) $\exists a \in \mathbf{Z}$ so that $\forall b \in \mathbf{Z}, (a-b)|(a+b)$.

(ii) $\forall a \in \mathbf{Z}^+ \ \exists b \in \mathbf{Z}^+$ so that $(a-b)|(a+b)$.

(iii) $\forall a \in \mathbf{Z}^+ \ \exists b \in \mathbf{Z}^+$ so that $(a+b)|(a-b)$.

(b) Write out the *negation* of the following statement:

$$\forall a, b \in \mathbf{Z}^+, \text{ if } a|2 \text{ and } b|3 \text{ then } (a+b)|5.$$

Then show that the negation is true, so that the original statement is false.

(c) Prove the following statement:

$$\exists N \in \mathbf{Z}^+ \text{ so that } \forall a, b \in \mathbf{Z}^+, \text{ if } a|2 \text{ and } b|3 \text{ then } (a+b)|N.$$

(a) (i) Choose $a = 0$. Then the statement to be proved is: $\forall b \in \mathbf{Z}, (-b)|b$. To prove this, let b be an arbitrary integer. Then $b = (-b)(-1)$ where -1 is an integer, so $(-b)|b$.

(ii) Let a be an arbitrary positive integer. We need to find a positive integer b (maybe depending on a) so that $(a-b)|(a+b)$. Choose $b = a+1$, which is a positive integer. Then $a - b = -1$ and $a + b = 2a + 1$, so we need to show that $(-1)|(2a+1)$. But this is clear, since $2a + 1 = (-1)(-2a-1)$ where $-2a-1$ is an integer.

(iii) Let a be an arbitrary positive integer. We need to find a positive integer b (maybe depending on a) so that $(a+b)|(a-b)$. Choose $b = a$, which is a positive integer. Then $a + b = 2a$ and $a - b = 0$, so we need to show that $(2a)|0$. But this is clear, since $0 = 0 \cdot 2a$.

(b) The negation is:

$$\exists a, b \in \mathbf{Z}^+ \text{ so that } a|2 \text{ and } b|3 \text{ but } (a+b) \not| 5.$$

This statement is true. For example we can choose $a = 1$ and $b = 1$; then $a|2$ and $b|3$ are both true, but $a + b = 2$, and $2 \not| 5$.

(c) For $a|2$ we need either $a = 1$ or $a = 2$, and for $b|3$ we need either $b = 1$ or $b = 3$. Thus we will need N to satisfy all of the following:

- $(1+1)|N$, which says $2|N$;
- $(2+1)|N$, which says $3|N$;
- $(1+3)|N$, which says $4|N$;
- $(2+3)|N$, which says $5|N$.

So for example, $N = 2 \cdot 3 \cdot 4 \cdot 5 = \mathbf{120}$ will work. Actually $N = 3 \cdot 4 \cdot 5 = \mathbf{60}$ will work too, and this is the smallest value of N which will work.

1. (a) Find all positive integers a so that $\lfloor a/271 \rfloor = 10$. How many such integers are there?
   (b) Find all positive integers a so that $\lfloor 271/a \rfloor = 10$.
   (c) Find all positive integers a so that $\lceil 271/a \rceil = 10$.
   (d) Prove or disprove: $\forall n \in \mathbf{Z}$, the equations $\lfloor 271/x \rfloor = n$ and $\lceil 271/x \rceil = n$ have the *same number* of integer solutions x.
   (e) Prove or disprove: $\exists n \in \mathbf{Z}$ so that $\forall a \in \mathbf{Z}$, $\lfloor 271/a \rfloor \neq n$.

   (a) For $\lfloor a/271 \rfloor = 10$ to be true we would need $10 \leq a/271 < 11$, or $2710 \leq a < 271 \cdot 11 = 2981$. So the values of a are $2710, 2711, 2712, \ldots, 2980$, a total of 271 integers.
   (b) Now we will need $10 \leq 271/a < 11$, or $10a \leq 271 < 11a$. This means $a \leq 271/10$ and $a > 271/11$, in other words $24.6 < a \leq 27.1$. So the allowed values of a are $25, 26, 27$.
   (c) Similarly, this time we will need $9 < 271/a \leq 10$, or $9a < 271 \leq 10a$. This means $a < 271/9$ and $a \geq 271/10$, in other words $27.1 \leq a < 30.1$. So the allowed values of a are $28, 29, 30$.
   (d) Despite the "evidence" from parts (b) and (c) (where there were 3 solutions each time), this statement is *false*. One counterexample is $n = 11$, as the only solutions for $\lfloor 271/x \rfloor = 11$ are $x = 23$ and 24, while $\lceil 271/x \rceil = 11$ has the three solutions $x = 25, 26$ and 27. Another counterexample is $n = 8$, since $\lfloor 271/x \rfloor = 8$ has three solutions $x = 31, 32$ and 33 while $\lceil 271/x \rceil = 8$ has the five solutions $x = 34$ to 38.

   An interesting counterexample is $n = 1$. Notice that the equation $\lceil 271/x \rceil = 1$ means $0 < 271/x \leq 1$, which is satisfied for *every* integer x greater than or equal to 271. So there are infinitely many solutions. But the equation $\lfloor 271/x \rfloor = 1$ means $1 \leq 271/x < 2$, and this inequality is satisfied only for the integers $x = 136, 137, \ldots, 271$.
   (e) This statement is *true*, and there are lots of integers n satisfying the condition. For example, any $n > 271$ will work, because $\lfloor 271/a \rfloor > 271$ is impossible for an integer a.

   Note. Can you find the *smallest* positive integer n for which this statement is true? If you think you have an answer to this question, talk to your professor or TA.

2. Let
$$S_n = \frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \frac{9}{4 \cdot 5} + \cdots - \frac{4n-3}{(2n-2)(2n-1)} + \frac{4n-1}{(2n-1)2n},$$
where the signs alternate.
   (a) Calculate and simplify S_1, S_2 and S_3.
   (b) Use part (a) (and more calculations if you need them) to guess a simple formula for S_n.
   (c) Prove your formula for all positive integers n *using mathematical induction.*
   (d) Give another proof of your formula for all positive integers n *using telescoping.* (See example 4.1.10 on page 205 of the text.)

(a) We get

$$S_1 = \frac{3}{1\cdot 2} = \frac{3}{2}\,, \qquad S_2 = \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} = \frac{3}{2} - \frac{5}{6} + \frac{7}{12} = \frac{18-10+7}{12} = \frac{15}{12} = \frac{5}{4}\,,$$

and (using our calculation for S_2)

$$S_3 = \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \frac{11}{5\cdot 6} = \frac{5}{4} - \frac{9}{20} + \frac{11}{30} = \frac{75-27+22}{60} = \frac{70}{60} = \frac{7}{6}\,.$$

(b) From the values in (a) we guess that $S_n = \dfrac{2n+1}{2n}$.

(c) *Basis step.* We need to prove that $S_1 = \dfrac{2\cdot 1+1}{2\cdot 1}$, which is true since both are 3/2.

Induction step. Assume that $S_k = \dfrac{2k+1}{2k}$ for some integer $k \geq 1$. We want to prove that $S_{k+1} = \dfrac{2(k+1)+1}{2(k+1)}$, which is the same as $S_{k+1} = \dfrac{2k+3}{2(k+1)}$. Well,

$$\begin{aligned}
S_{k+1} &= \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \cdots + \frac{4k-1}{(2k-1)2k} - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \\
&= S_k - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \\
&= \frac{2k+1}{2k} - \frac{4k+1}{2k(2k+1)} + \frac{4k+3}{(2k+1)(2k+2)} \qquad \text{by our assumption} \\
&= \frac{(2k+1)^2(2k+2) - (4k+1)(2k+2) + (4k+3)2k}{2k(2k+1)(2k+2)} \\
&= \frac{[4k^2+4k+1-(4k+1)](2k+2) + (8k^2+6k)}{2k(2k+1)(2k+2)} \\
&= \frac{4k^2(2k+2) + (8k^2+6k)}{2k(2k+1)(2k+2)} = \frac{8k^3+16k^2+6k}{2k(2k+1)(2k+2)} \\
&= \frac{2k(2k+1)(2k+3)}{2k(2k+1)(2k+2)} = \frac{2k+3}{2k+2}\,,
\end{aligned}$$

which proves the induction step.

Therefore the statement is true for all integers $n \geq 1$.

(d) Notice that

$$\frac{3}{1\cdot 2} = \frac{1}{1} + \frac{1}{2}\,, \qquad \frac{5}{2\cdot 3} = \frac{1}{2} + \frac{1}{3}\,, \qquad \frac{7}{3\cdot 4} = \frac{1}{3} + \frac{1}{4}\,,$$

and in general

$$\frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$$

for any positive integer k. Thus

$$
\begin{aligned}
S_n &= \frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \cdots - \frac{4n-3}{(2n-2)(2n-1)} + \frac{4n-1}{(2n-1)2n} \\
&= \left(\frac{1}{1}+\frac{1}{2}\right) - \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) - \cdots - \left(\frac{1}{2n-2}+\frac{1}{2n-1}\right) + \left(\frac{1}{2n-1}+\frac{1}{2n}\right) \\
&= \frac{1}{1} + \frac{1}{2n} = \frac{2n+1}{2n}\ ,
\end{aligned}
$$

so our guess is proved.

3. (a) Prove the following statement by contradiction: for all integers n, if $3|n$ then $3 \not| (n+271)$.
   (b) Prove or disprove: for all integers n, if $3|n$ then $5 \not| n$.
   (c) Prove by mathematical induction that $3|(2^n - (-1)^n)$ for all integers $n \geq 1$.

   (a) Assume that $3|n$ for some integer n. This means that $n = 3k$ for some integer k. We want to prove that $3 \not| (n+271)$. To get a proof by contradiction, we assume that what we want to prove is false: namely we will assume that $3|(n+271)$. This means we are also assuming that $n + 271 = 3\ell$ for some integer ℓ. Now our two assumptions tell us that

$$271 = (n + 271) - n = 3\ell - 3k = 3(\ell - k),$$

   where $\ell - k$ is an integer. Thus $3|271$, which however is false. Thus our assumption that $3|(n + 271)$ must be false, so $3 \not| (n + 271)$.

   (b) This is *false*. A counterexample is $n = 15$, since $3|15$ but also $5|15$. Another counterexample is $n = 0$.

   (c) *Basis step.* We need to prove that $3|(2^1 - (-1)^1)$, which says $3|(2 + 1)$ or $3|3$. This is true.

   Induction step. Assume that $3|(2^k - (-1)^k)$ for some integer $k \geq 1$. This means that $2^k - (-1)^k = 3\ell$ for some integer ℓ. We want to prove that $3|(2^{k+1} - (-1)^{k+1})$. Well,

$$
\begin{aligned}
2^{k+1} - (-1)^{k+1} &= 2\cdot 2^k - (-1)\cdot(-1)^k \\
&= 2(2^k - (-1)^k) + 2(-1)^k + (-1)^k \\
&= 2(3\ell) + 3(-1)^k \quad \text{by our assumption} \\
&= 3(2\ell + (-1)^k),
\end{aligned}
$$

   where $2\ell + (-1)^k$ is an integer. Thus $3|(2^{k+1} - (-1)^{k+1})$.
   Therefore, by induction, $3|(2^n - (-1)^n)$ for all integers $n \geq 1$.

MATH 271 ASSIGNMENT 3 SOLUTIONS

1. (a) Prove **by induction** that, for all integers $n \geq 2$,

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{n^2}{(n+1)!} \leq 2 - \frac{2n}{(n+1)!} \ . \tag{1}$$

(b) Prove that in fact inequality (1) holds for all integers $n \geq 1$.

(c) Find the smallest real number A so that, for all integers $n \geq 1$,

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{n^2}{(n+1)!} \leq A - \frac{2n}{(n+1)!} \ .$$

(a) *Basis step.* When $n = 2$ inequality (1) is

$$\frac{1^2}{2!} + \frac{2^2}{3!} \leq 2 - \frac{4}{3!}$$

which is

$$\frac{1}{2} + \frac{4}{6} \leq 2 - \frac{4}{6} \ , \quad \text{that is} \quad \frac{7}{6} \leq \frac{8}{6} \ ,$$

which is true.

Inductive step. Assume that inequality (1) holds for some integer $n = k$, where $k \geq 2$. We want to prove that inequality (1) holds for $n = k + 1$. So we are assuming that

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{k^2}{(k+1)!} \leq 2 - \frac{2k}{(k+1)!} \ ,$$

and we want to prove that

$$\frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \cdots + \frac{(k+1)^2}{(k+2)!} \leq 2 - \frac{2(k+1)}{(k+2)!} \ . \tag{2}$$

Well,

$$\begin{aligned}
\frac{1^2}{2!} + \frac{2^2}{3!} + \cdots + \frac{(k+1)^2}{(k+2)!} &= \frac{1^2}{2!} + \frac{2^2}{3!} + \cdots + \frac{k^2}{(k+1)!} + \frac{(k+1)^2}{(k+2)!} \\
&\leq 2 - \frac{2k}{(k+1)!} + \frac{(k+1)^2}{(k+2)!} \quad \text{by our assumption} \\
&= 2 - \frac{2k(k+2) - (k+1)^2}{(k+2)!} \\
&= 2 - \frac{2k^2 + 4k - k^2 - 2k - 1}{(k+2)!} \\
&= 2 - \frac{k^2 + 2k - 1}{(k+2)!} \ .
\end{aligned}$$

So in order to prove (2), we would like to prove that

$$2 - \frac{k^2 + 2k - 1}{(k+2)!} \leq 2 - \frac{2(k+1)}{(k+2)!} \ .$$

This is equivalent successively to

$$-\frac{k^2 + 2k - 1}{(k+2)!} \leq -\frac{2(k+1)}{(k+2)!} \ ,$$

$$\frac{k^2 + 2k - 1}{(k+2)!} \geq \frac{2(k+1)}{(k+2)!} \ ,$$

and thus to

$$k^2 + 2k - 1 \geq 2k + 2, \quad \text{that is,} \quad k^2 \geq 3,$$

which is true since $k \geq 2$. This finishes the proof of the inductive step. Thus inequality (1) holds for all integers $n \geq 2$.

(b) When $n = 1$, inequality (1) says

$$\frac{1^2}{2!} \leq 2 - \frac{2}{2!}$$

which is $1/2 \leq 1$, which is true. Since in part (a) we proved that inequality (1) holds for all integers $n \geq 2$, we now know it holds for all integers $n \geq 1$. Notice that, since the inductive step needed that $k \geq 2$, to prove inequality (1) for all $n \geq 1$ we need both cases $n = 1$ and $n = 2$ in the basis step.

(c) The inductive step in the proof in part (a) works just the same if the 2 right after the inequality sign is replaced with any number A. So the inequality in part (c) will hold for all integers $n \geq 1$ provided that it holds for $n = 1$ and $n = 2$, which is the basis step. When $n = 1$ the inequality in (c) says

$$\frac{1^2}{2!} \leq A - \frac{2}{2!}$$

which simplifies to $A \geq 3/2$. When $n = 2$ the inequality in (c) says

$$\frac{1^2}{2!} + \frac{2^2}{3!} \leq A - \frac{4}{3!}$$

which simplifies to $A \geq 1/2 + 4/6 + 4/6 = 11/6$. We need both of these to hold, so the smallest A that will work is $A = \mathbf{11/6}$.

2. You are given the following "while" loop:

   [*Pre-condition*: m is a nonnegative integer, $a = 0$, $b = 1$, $c = 2$, $i = 0$.]

   while $(i \neq m)$

   1. $a := b$
   2. $b := c$
   3. $c := 2b - a$
   4. $i := i + 1$

   end while

   [*Post-condition*: $c = m + 2$.]

   Loop invariant: $I(n)$ is "$a = n$, $b = n + 1$, $c = n + 2$, $i = n$".

   (a) Prove the correctness of this loop with respect to the pre- and post-conditions.

   (b) Suppose the "while" loop is as above, except that the pre-condition is replaced by: m is a nonnegative integer, $a = 1$, $b = 3$, $c = 5$, $i = 0$. Find a post-condition that gives the final value of c, and an appropriate loop invariant, and prove the correctness of this loop.

   (a) We first need to check that the loop invariant holds when $n = 0$. $I(0)$ says $a = 0$, $b = 1$, $c = 2$ and $i = 0$, and these are all true by the pre-conditions.

   So now assume that the loop invariant $I(k)$ holds for some integer $k \geq 0$, $k < m$. We want to prove that $I(k + 1)$ holds, that is, that the loop invariant will still hold after one more pass through the loop. So we are assuming that $a = k$, $b = k + 1$, $c = k + 2$ and $i = k$, and we now go through the loop. Step 1 sets a equal to $b = k + 1$, then step 2 sets b equal to $c = k+2$, then step 3 sets c equal to $2b - a = 2(k+2) - (k+1) = k+3$, then step 4 sets i equal to $k + 1$. This means that $I(k + 1)$ is true, as required.

   Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $c = m + 2$ as required.

   (b) If we set the variables to their pre-condition values of $a = 1$, $b = 3$, $c = 5$ and $i = 0$, and run through the loop, the new values we get are $a = 3$, $b = 5$, $c = 2(5) - 3 = 7$, and $i = 1$. From this (or by running through the loop once or twice more to collect more evidence) we can guess that the loop invariant we want will be

   $$I(n) : a = 2n + 1, b = 2n + 3, c = 2n + 5, i = n,$$

   and the post-condition value of c ought to be $c = 2m + 5$. This choice of $I(n)$ becomes $a = 1$, $b = 3$, $c = 5$ and $i = 0$ when $n = 0$, so the pre-condition is satisfied.

   So now we assume that the new loop invariant $I(k)$ holds for some integer $k \geq 0$, $k < m$, and we want to prove that $I(k+1)$ holds. So we are assuming that $a = 2k + 1$, $b = 2k + 3$, $c = 2k + 5$ and $i = k$, and we now go through the loop. Step 1 sets a equal to $b = 2k + 3 = 2(k + 1) + 1$, then step 2 sets b equal to $c = 2k + 5 = 2(k + 1) + 3$, then step 3 sets c equal to $2b - a = 2(2k + 5) - (2k + 3) = 2k + 7 = 2(k + 1) + 5$, then step 4 sets i equal to $k + 1$. This means that $I(k + 1)$ is true, as required.

Finally the loop stops when $i = m$, and we need to check that at that point the post-condition is satisfied. When $i = m$ it means that the loop invariant $I(m)$ must hold, so from $I(m)$ we know that $c = 2m + 5$ as required.

3. Prove or disprove each of the following six statements. Proofs should use the "element" methods given in Section 5.2. [*Note*: $\mathcal{P}(X)$ denotes the power set of the set X.]

   (a) For all sets A, B, C, $(A - B) \times C \subseteq (A \times C) - (B \times C)$.

   (b) For all sets A, B, C, $(A \times C) - (B \times C) \subseteq (A - B) \times C$.

   (c) For all sets A, B, C, $(A - B) \times C = (A \times C) - (B \times C)$.

   (d) For all sets A and B, $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   (e) For all sets A and B, $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   (f) For all sets A and B, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

   (a) This inequality is **true**. Here is a proof.

   Let A, B, C be arbitrary sets. Note that the left side of this inequality is a Cartesian product, which means that its elements will be ordered pairs. So let (a, c) be an arbitrary element of $(A - B) \times C$. This means that $a \in A - B$ and $c \in C$. Since $a \in A - B$, this means that $a \in A$ and $a \notin B$. Since $a \in A$ and $c \in C$, we get that $(a, c) \in A \times C$. But since $a \notin B$, we know that (a, c) cannot be an element of $B \times C$. Since $(a, c) \in A \times C$ but $(a, c) \notin B \times C$, we know $(a, c) \in (A \times C) - (B \times C)$. Therefore $(A - B) \times C \subseteq (A \times C) - (B \times C)$.

   (b) Similarly, this inequality is **true**, and we can reverse our steps in part (a) to get a proof.

   Let (a, c) be an arbitrary element of $(A \times C) - (B \times C)$. This means that $(a, c) \in A \times C$ but $(a, c) \notin B \times C$. Since $(a, c) \in A \times C$, we know that $a \in A$ and $c \in C$. But since $(a, c) \notin B \times C$ although $c \in C$, we also know $a \notin B$. Thus $a \in A$ and $a \notin B$, which means $a \in A - B$. Thus $(a, c) \in (A - B) \times C$. Therefore $(A \times C) - (B \times C) \subseteq (A - B) \times C$.

   (c) Since the inequalities in parts (a) and (b) both hold, we get that the equality in (c) holds for all sets A, B, C.

   (d) This inequality is **false** no matter what sets we choose for A and B! To see this, let A and B be any sets. Notice that the empty set $\emptyset \subseteq A - B$ regardless of what A and B are, so $\emptyset \in \mathcal{P}(A - B)$. However, since $\emptyset \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(B)$, we get $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$. Therefore $\mathcal{P}(A - B) \not\subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   Note. You can prove that if X is any *nonempty* set so that $X \in \mathcal{P}(A - B)$, then $X \in \mathcal{P}(A) - \mathcal{P}(B)$. So the only counterexample to the inequality in part (d) is the empty set.

   (e) This inequality is also **false**, but counterexamples are easier to find. For example, let $A = \{1, 2\}$ and $B = \{1\}$. Then $\{1, 2\} \subseteq A$ and $\{1, 2\} \not\subseteq B$, so $\{1, 2\} \in \mathcal{P}(A)$ and $\{1, 2\} \notin \mathcal{P}(B)$, so $\{1, 2\} \in \mathcal{P}(A) - \mathcal{P}(B)$. However $A - B = \{2\}$, so $\{1, 2\} \notin \mathcal{P}(A - B)$. Therefore $\mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B)$.

   (f) Since the inequality in (e) (or (d)) fails, the equality in (f) fails too.

1. For each positive integer n, let $[n] = \{1, 2, 3, \ldots, n\}$, and define

$$\begin{aligned}\mathcal{S}_{\cup}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of sets such that } A \cup B = [n];\\ \mathcal{S}_{\cap}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \cap B = \emptyset;\\ \mathcal{S}_{\subseteq}(n) &= \text{ the set of all ordered pairs } (A, B) \text{ of subsets of } [n] \text{ such that } A \subseteq B.\end{aligned}$$

(a) Find $\mathcal{S}_{\cup}(1)$ and $\mathcal{S}_{\cup}(2)$.

(b) Prove that $\mathcal{S}_{\cup}(n)$ has exactly 3^n elements.

(c) Prove that $(A, B) \in \mathcal{S}_{\cup}(n)$ if and only if $(A^c, B^c) \in \mathcal{S}_{\cap}(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_{\cap}(n)$.

(d) Prove that $(A, B) \in \mathcal{S}_{\cup}(n)$ if and only if $(A^c, B) \in \mathcal{S}_{\subseteq}(n)$ (here $[n]$ is the universal set). Therefore find the number of elements in $\mathcal{S}_{\subseteq}(n)$.

(a) We get

$$\mathcal{S}_{\cup}(1) = \big\{(\{1\}, \emptyset),\ (\emptyset, \{1\}),\ (\{1\}, \{1\})\big\}$$

and

$$\mathcal{S}_{\cup}(2) = \Big\{(\{1,2\}, \emptyset),\ (\emptyset, \{1,2\}),\ (\{1,2\}, \{1\}),\ (\{1\}, \{1,2\}),\ (\{1,2\}, \{2\}),\ (\{2\}, \{1,2\}),\ (\{1,2\}, \{1,2\}),\ (\{1\}, \{2\}),\ (\{2\}, \{1\})\Big\}.$$

(b) We count how many ways there are to construct sets A and B so that $A \cup B = \{1, 2, \ldots, n\}$. To get this union, we need each number from 1 to n to either be in A, or in B, or in both. So we have three possibilities for each of the n numbers from 1 to n. Since these choices are all independent, there are $3 \cdot 3 \cdot \ldots \cdot 3 = 3^n$ such ordered pairs (A, B).

(c) First assume that $(A, B) \in \mathcal{S}_{\cup}(n)$. Then $A \cup B = [n]$, so by De Morgan's Law (page 272, #9(a)),

$$A^c \cap B^c = (A \cup B)^c = [n]^c = \emptyset,$$

therefore $(A^c, B^c) \in \mathcal{S}_{\cap}(n)$.

Conversely, assume that $(A^c, B^c) \in \mathcal{S}_{\cap}(n)$. Then $A^c \cap B^c = \emptyset$, so by various properties on page 272,

$$A \cup B = (A^c)^c \cup (B^c)^c = (A^c \cap B^c)^c = \emptyset^c = [n],$$

therefore $(A, B) \in \mathcal{S}_{\cup}(n)$.

This means that there is a one-to-one correspondence between the elements of $\mathcal{S}_{\cup}(n)$ and the elements of $\mathcal{S}_{\cap}(n)$, so by part (b) $\mathcal{S}_{\cap}(n)$ must also have 3^n elements.

(d) First assume that $(A, B) \in \mathcal{S}_{\cup}(n)$, which means $A \cup B = [n]$. We want to prove that $(A^c, B) \in \mathcal{S}_{\subseteq}(n)$, which means we want to prove that $A^c \subseteq B$. Let $x \in A^c$ be arbitrary. This means that $x \in [n]$ but $x \notin A$. Since $A \cup B = [n]$, $x \in [n]$ means $x \in A \cup B$, and since $x \notin A$ we conclude that $x \in B$. Therefore $A^c \subseteq B$ and $(A^c, B) \in \mathcal{S}_{\subseteq}(n)$.

Conversely, assume that $(A^c, B) \in \mathcal{S}_{\subseteq}(n)$, which means $A^c \subseteq B$. We want to prove that $(A, B) \in \mathcal{S}_{\cup}(n)$, which means we want to prove that $A \cup B = [n]$. Since $A \cup B \subseteq [n]$, we only need to prove that $[n] \subseteq A \cup B$. Let $x \in [n]$ be arbitrary. If $x \in A$, then $x \in A \cup B$ which is what we want. On the other hand, if $x \notin A$, then $x \in A^c$, and since $A^c \subseteq B$, this means that $x \in B$ and thus $x \in A \cup B$. So in either case we get that $x \in A \cup B$. Therefore $[n] \subseteq A \cup B$, so $A \cup B = [n]$, so $(A, B) \in \mathcal{S}_{\cup}(n)$.

Once again this means that there is a one-to-one correspondence between the elements of $\mathcal{S}_{\cup}(n)$ and the elements of $\mathcal{S}_{\subseteq}(n)$, so by part (b) $\mathcal{S}_{\subseteq}(n)$ must also have 3^n elements.

2. For each positive integer n, let $f(n)$ be the number of ordered pairs (A, B) of subsets of $\{1, 2, 3, \ldots, n\}$ so that $A \cup B$ has an even number of elements.

   (a) Find $f(1)$ and $f(2)$ by listing all the ordered pairs of subsets.

   (b) Use Problem 1(b) to prove that for any n,

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k}.$$

   Show that your answers to part (a) agree with this formula.

   (c) Mimic Example 6.7.4 on page 368 to prove that $\sum_{i=0}^{n} \binom{n}{i} 3^i = 4^n$ and thus

$$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n - f(n).$$

   (d) Use Pascal's Formula (page 360), (b) and (c), and mathematical induction to prove that

$$f(n) = \begin{cases} 2^{n-1}(2^n - 1) & \text{if } n \text{ is odd,} \\ 2^{n-1}(2^n + 1) & \text{if } n \text{ is even.} \end{cases}$$

(a) Since A and B are subsets of $\{1, 2, \ldots, n\}$, we always have $A \cup B \subseteq \{1, 2, \ldots, n\}$. So when $n = 1$, the only way for $A \cup B$ to have an even number of elements is if $A \cup B = \emptyset$, so the only ordered pair (A, B) that works is $(\emptyset, \emptyset)$, and thus $f(1) = \mathbf{1}$. When $n = 2$, we could have $A \cup B = \emptyset$ or $A \cup B = \{1, 2\}$, so the ordered pairs (A, B) that work are $(\emptyset, \emptyset)$ plus the nine ordered pairs in $\mathcal{S}_{\cup}(2)$ from problem 1(a). Thus $f(2) = \mathbf{10}$.

(b) First, from problem 1(b) it is clear that for any set S with m elements there must be exactly 3^m ordered pairs (A, B) of sets so that $A \cup B = S$ (since the names of the m elements of S don't matter). Let k be an integer so that $0 \le 2k \le n$. There are $\binom{n}{2k}$ subsets of $\{1, 2, \ldots, n\}$ with $2k$ elements, and for each of these subsets there are 3^{2k} ordered pairs (A, B) of sets whose union is that subset. Thus for each k, there are $\binom{n}{2k} 3^{2k}$ ordered pairs (A, B) of subsets of $\{1, 2, \ldots, n\}$ so that $A \cup B$ has $2k$ elements. Adding over all possible values of k (namely $k = 0, 1, \ldots, \lfloor n/2 \rfloor$), we get that

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k}.$$

When $n = 1$ this says

$$f(1) = \sum_{k=0}^{0} \binom{1}{2k} 3^{2k} = \binom{1}{0} 3^0 = 1,$$

and when $n = 2$ it says

$$f(2) = \sum_{k=0}^{1} \binom{2}{2k} 3^{2k} = \binom{2}{0} 3^0 + \binom{2}{2} 3^2 = 1 + 9 = 10,$$

both agreeing with part (a).

(c) We put $a = 1$ and $b = 3$ into the Binomial Theorem (Theorem 6.7.1 on page 364) to get

$$\sum_{i=0}^{n} \binom{n}{i} 3^i = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 3^i = (1+3)^n = 4^n.$$

Splitting this sum into two parts, one with all the even i's and one with all the odd i's, we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n.$$

But the first sum is just $f(n)$ by part (b), so subtracting it from both sides gives us

$$\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3^{2k-1} = 4^n - f(n)$$

as required.

(d) *Basis step.* When $n = 1$ (which is odd) the formula says $f(1) = 2^0(2^1 - 1) = 1$, which is correct by part (a).

Inductive step. Assume that the formula is correct for some integer $n \geq 1$. We want to prove it is correct for the next integer $n + 1$. Well,

$$\begin{aligned}
f(n+1) &= \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} 3^{2k} \qquad \text{by part (b)} \\
&= \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \left[\binom{n}{2k} + \binom{n}{2k-1} \right] 3^{2k} \qquad \text{by Pascal's Formula} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{2k} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2k-1} 3 \cdot 3^{2k-1} \\
&= f(n) + 3(4^n - f(n)) \qquad \text{by parts (b) and (c)} \\
&= 3(4^n) - 2f(n) \\
&= \begin{cases} 3(4^n) - 2^n(2^n - 1) & \text{if } n \text{ is odd} \\ 3(4^n) - 2^n(2^n + 1) & \text{if } n \text{ is even} \end{cases} \qquad \text{by assumption} \\
&= \begin{cases} 2(4^n) + 2^n = 2^n(2^{n+1} + 1) & \text{if } n+1 \text{ is even} \\ 2(4^n) - 2^n = 2^n(2^{n+1} - 1) & \text{if } n+1 \text{ is odd,} \end{cases}
\end{aligned}$$

which completes the inductive step. Therefore the formula is correct for all integers $n \geq 1$.

Note: If n is odd, and if $2^n - 1$ happens to be a prime number, then the value $f(n) = 2^{n-1}(2^n - 1)$ is what is called a *perfect number*. To find out what these are, ask your professor or TA, or search the internet.

3. Again let $[n] = \{1, 2, 3, \ldots, n\}$ for any positive integer n.

   (a) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (b) Find the number of one-to-one functions $f : [n] \to [n]$ such that $f(k) \leq k \ \forall k \in [n]$.

   (c) Find the number of functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.

   (d) Find the number of onto functions $f : [n] \to [n]$ such that $f(k) \leq k + 1 \ \forall k \in [n]$.

   (a) Since, for every k, $f(k)$ must be one of the k values $1, 2, \ldots, k$, there is one choice for $f(1)$ (namely 1), two choices for $f(2)$ (namely 1 or 2), and so on up to n choices for $f(n)$ (namely any of $1, 2, \ldots, n$). Thus by the Multiplication Rule there are $1 \cdot 2 \cdot \ldots \cdot n = n!$ ways to assign all the values $f(1), f(2), \ldots, f(n)$, that is, $\mathbf{n!}$ different functions.

   (b) If f must be one-to-one, then we still must assign $f(1) = 1$, but then we cannot assign $f(2)$ to be 1 too, so we must put $f(2) = 2$. Next we cannot let $f(3)$ be 1 or 2, so we must put $f(3) = 3$. Continuing in this way, we are forced to put $f(k) = k$ for each k, so there is just **one** one-to-one function $f : [n] \to [n]$, namely the identity function.

   (c) Proceeding as in part (a), for each k, $f(k)$ must be one of the $k+1$ choices $1, 2, \ldots, k+1$, provided that $k < n$. So $f(1)$ can be 1 or 2, $f(2)$ can be 1, 2 or 3, and so on up to $f(n-1)$ which can be any of $1, 2, \ldots, n$. But $f(n)$ must still belong to $[n]$ so there are only n choices for $f(n)$. Thus by the Multiplication Rule the total number of functions is $2 \cdot 3 \cdot \ldots \cdot n \cdot n = \mathbf{n(n!)}$.

   (d) Note that since $[n]$ is finite, a function $f : [n] \to [n]$ is onto if and only if it is one-to-one. So we are really just counting one-to-one functions again. Now $f(1)$ must be 1 or 2, so there are two choices for $f(1)$. Then $f(2)$ must be 1, 2 or 3, so removing whichever choice we made for $f(1)$ will leave two choices for $f(2)$. In general there will be $k+1$ choices for $f(k)$ (namely $1, 2, \ldots, k+1$), but after we remove the choices we make for $f(1), f(2), \ldots, f(k-1)$ we will always have exactly two choices left for $f(k+1)$. The exception again is that for $f(n)$ there are only n choices originally (namely $1, 2, \ldots, n$), and after we remove the choices we make for $f(1), f(2), \ldots, f(n-1)$ we will only have one choice left for $f(n)$. So in total there will be $2 \cdot 2 \cdot \ldots \cdot 2 \cdot 1 = \mathbf{2^{n-1}}$ onto functions.

1. If $f : \mathbf{R} \to \mathbf{R}$ is a function (where $\mathbf{R}$ is the set of all real numbers), we define the function $f^{(2)}$ to be the composition $f \circ f$, and for any integer $n \geq 2$, define $f^{(n+1)} = f \circ f^{(n)}$. So $f^{(2)}(x) = (f \circ f)(x) = f(f(x))$, $f^{(3)}(x) = (f \circ f^{(2)})(x) = f(f(f(x)))$, and so on.

   Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x^2$ for all $x \in \mathbf{R}$.

   (a) Find and simplify $f^{(2)}(x)$ and $f^{(3)}(x)$.

   (b) Use part (a) (and more calculations if you need them) to guess a formula for $f^{(n)}(x)$.

   (c) Prove your guess using mathematical induction.

   (d) Find all $x \in \mathbf{R}$ so that $f^{(271)}(x) = x$.

   (a) We get
$$f^{(2)}(x) = f(f(x)) = f(3x^2) = 3(3x^2)^2 = 3^3x^4$$
   and
$$f^{(3)}(x) = f(f^{(2)}(x)) = f(3^3x^4) = 3(3^3x^4)^2 = 3^7x^8.$$

   (b) Since $f^{(2)}(x) = 3^3x^4 = 3^{2^2-1}x^{2^2}$ and $f^{(3)}(x) = 3^7x^8 = 3^{2^3-1}x^{2^3}$, we guess that
$$f^{(n)}(x) = 3^{2^n-1}x^{2^n} \qquad \text{for all } n \geq 2.$$

   (c) *Basis step.* This is already done, since our formula is true when $n = 2$.
   Inductive step. Assume that $f^{(k)}(x) = 3^{2^k-1}x^{2^k}$ for some integer $k \geq 2$. We want to prove that $f^{(k+1)}(x) = 3^{2^{k+1}-1}x^{2^{k+1}}$. Well,
$$\begin{aligned} f^{(k+1)}(x) &= f\left(f^{(k)}(x)\right) \quad \text{by definition} \\ &= f\left(3^{2^k-1}x^{2^k}\right) \quad \text{by assumption} \\ &= 3\left(3^{2^k-1}x^{2^k}\right)^2 \\ &= 3^{1+(2^k-1)2}x^{(2^k)2} \\ &= 3^{2^{k+1}-1}x^{2^{k+1}}, \end{aligned}$$
   which finishes the inductive step. This proves that our guess is correct for every integer $n \geq 2$.

   (d) Since $f^{(271)}(x) = 3^{2^{271}-1}x^{2^{271}}$, we need to solve the equation $3^{2^{271}-1}x^{2^{271}} = x$. One obvious solution is $x = \mathbf{0}$. So assuming now that $x \neq 0$, we can divide both sides by x and get $3^{2^{271}-1}x^{2^{271}-1} = 1$, which can be rewritten as $(3x)^{2^{271}-1} = 1$, which means that $3x = 1$ since $2^{271} - 1$ is odd. Thus the only other solution is $x = \mathbf{1/3}$.

2. Let $[n] = \{1, 2, \ldots, n\}$, where n is a positive integer. Let $\mathcal{R}$ be the relation on the power set $\mathcal{P}([n])$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{R}B$ if and only if $1 \notin A - B$.

   (a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Explain.

(b) Find the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ such that $A\mathcal{R}B$. [*Hint*: first count the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ so that $A \not\mathcal{R} B$.]

(c) Suppose you choose sets $A, B \in \mathcal{P}([n])$ at random. What is the probability that $A\mathcal{R}B$?

(d) Let $\mathcal{S}$ be the relation on the power set $\mathcal{P}([n])$ defined by: for $A, B \in \mathcal{P}([n])$, $A\mathcal{S}B$ if and only if $1 \in A - B$. Is $\mathcal{S}$ transitive? Explain.

(a) $\mathcal{R}$ **is reflexive**. Here is a proof. Let $A \in \mathcal{P}([n])$ be arbitrary. Then $A - A = \emptyset$, so $1 \notin A - A$, so $A\mathcal{R}A$.

$\mathcal{R}$ **is not symmetric**. Here is a counterexample. Let $A = \emptyset$ and $B = \{1\}$. Then $A - B = \emptyset$, so $1 \notin A - B$, so $A\mathcal{R}B$. However $B - A = \{1\}$, so $1 \in B - A$, so $B \not\mathcal{R} A$.

$\mathcal{R}$ **is transitive**. Here is a proof. Let $A, B, C \in \mathcal{P}([n])$ be arbitrary so that $A\mathcal{R}B$ and $B\mathcal{R}C$. This means that $1 \notin A - B$ and $1 \notin B - C$. We want to prove that $A\mathcal{R}C$, which means we want to prove that $1 \notin A - C$. We do this by contradiction. Suppose that $1 \in A - C$. This means that $1 \in A$ but $1 \notin C$. Since $1 \in A$ but $1 \notin A - B$, it must mean that $1 \in B$. But now $1 \in B$ and $1 \notin C$ means $1 \in B - C$, which is a contradiction. Therefore $1 \notin A - C$, so $\mathcal{R}$ is transitive.

(b) Since $A \not\mathcal{R} B$ means $1 \in A - B$, to count the number of ordered pairs (A, B) so that $A \not\mathcal{R} B$ we just count the number of (A, B) so that $1 \in A$ and $1 \notin B$. The number of subsets A of $[n]$ so that $1 \in A$ is just the number of subsets of $\{2, 3, \ldots, n\}$, which is 2^{n-1} (for example, see p. 285). The number of subsets B of $[n]$ so that $1 \notin B$ is also just the number of subsets of $\{2, 3, \ldots, n\}$, which is 2^{n-1}. Thus the number of ordered pairs (A, B) so that $A \not\mathcal{R} B$ is $2^{n-1} \cdot 2^{n-1} = 2^{2(n-1)}$ by the Multiplication Rule. There are 2^n subsets of $[n]$ altogether, so there are $2^n \cdot 2^n = 2^{2n}$ ordered pairs (A, B) altogether. Therefore the number of ordered pairs (A, B) of sets in $\mathcal{P}([n])$ such that $A\mathcal{R}B$ is $2^{2n} - 2^{2(n-1)} = 2^{2n-2}(2^2 - 1) = 3(2^{2n-2})$.

(c) Since all choices of subsets $A, B \in \mathcal{P}([n])$ are equally likely, the probability is

$$\frac{\text{number of } (A, B) \text{ so that } A\mathcal{R}B}{\text{total number of } (A, B)} = \frac{3(2^{2n-2})}{2^{2n}} = \frac{3}{4},$$

regardless of the value of n.

(d) Yes, $\mathcal{S}$ is transitive, **vacuously**. Suppose that $A, B, C \in \mathcal{P}([n])$ satisfy $A\mathcal{S}B$ and $B\mathcal{S}C$. This means that $1 \in A - B$ and $1 \in B - C$. But $1 \in A - B$ means in particular that $1 \notin B$, while $1 \in B - C$ means in particular that $1 \in B$. This is a contradiction, so the "if" part of the definition of transitivity can never happen, so the relation $\mathcal{S}$ is transitive vacuously.

3. Let $\mathcal{F}$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, where n is a positive integer. Define a relation R on $\mathcal{F}$ by: for $f, g \in \mathcal{F}$, fRg if and only if $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$.

(a) Prove that R is an equivalence relation on $\mathcal{F}$.

(b) Suppose that $n = 2m + 1$ is odd. Find the number of functions in the equivalence class $[id]$, where id is the identity function on $\{1, 2, \ldots, n\}$. How many of these functions are one-to-one and onto?

(c) Suppose that $n = 2m$ is even. Find the number of functions in the equivalence class $[g]$, where $g(x) = 1$ is a constant function. How many of these functions are one-to-one and onto?

(a) *R is reflexive.* Let $f \in \mathcal{F}$ be arbitrary. Then $f(k) + f(k) = 2f(k)$ is even for every $k \in \{1, 2, \ldots, n\}$, since $f(k)$ is an integer, so fRf.

R is symmetric. Let $f, g \in \mathcal{F}$ be arbitrary so that fRg. This means that $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$. But then $g(k) + f(k) = f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, so gRf.

R is transitive. Let $f, g, h \in \mathcal{F}$ be arbitrary so that fRg and gRh. This means that $f(k) + g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, and $g(k) + h(k)$ is even for all $k \in \{1, 2, \ldots, n\}$. But then $f(k) + g(k) + g(k) + h(k) = f(k) + h(k) + 2g(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, so $f(k) + h(k)$ is even for all $k \in \{1, 2, \ldots, n\}$, since the sum and difference of even numbers is even. Therefore fRh.

(b) We want to count the number of functions $f \in \mathcal{F}$ so that $fR\ id$. id is the function $id(k) = k$ for all $k \in \{1, 2, \ldots, n\}$. Thus $fR\ id$ means that $f(k) + k$ is even for all k. This in turn means that $f(k)$ must be even whenever k is even, and odd whenever k is odd. Since $n = 2m + 1$, there are m even numbers and $m + 1$ odd numbers in $\{1, 2, \ldots, n\}$. So for each of the m even k's there are m choices for $f(k)$, and for each of the $m + 1$ odd k's there are $m + 1$ choices for $f(k)$. Thus the total number of ways we can define f is $\mathbf{m^m(m+1)^{m+1}}$.

If we insist that f be one-to-one and onto, we only need to make it one-to-one, since any one-to-one function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ will have to be onto as well. Now when we count how many ways there are to define $f(2), f(4), \ldots, f(2m)$ (that is, $f(k)$ for the m even k's), we get m ways to define $f(2)$, $m - 1$ ways to define $f(4)$, and so on down to just one way to define $f(2m)$. So there are $m!$ ways to define $f(2), f(4), \ldots, f(2m)$. Similarly, there are $m + 1$ ways to define $f(1)$, m ways to define $f(3)$, and so on down to just one way to define $f(2m+1)$. So there are $(m+1)!$ ways to define $f(1), f(3), \ldots, f(2m + 1)$. Thus altogether there are $\mathbf{m!(m+1)!}$ ways to define f so that it is one-to-one and onto.

(c) This time we want to count the number of functions $f \in \mathcal{F}$ so that fRg. But since $g(k) = 1$ for all $k \in \{1, 2, \ldots, n\}$, to get $f(k) + g(k)$ to be even for all k, we will need that $f(k)$ is odd for all k. Since $n = 2m$, there are m odd numbers in $\{1, 2, \ldots, n\}$. So we have m choices for each $f(k)$, and thus the total number of ways we can define f is $m^n = \mathbf{m^{2m}}$.

If we insist that f be one-to-one and onto, once again we only need to make it one-to-one. But since $f(k)$ must be odd for every k, and the total number of k's ($2m$) is greater than the number of odd numbers available (m), it is impossible to assign a different odd number to each $f(k)$. Thus the number of one-to-one onto functions this time is **zero**.