MATH 271 ASSIGNMENT 1 SOLUTIONS

1. For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.

   (a) $\forall x \in \mathbb{Q}$, x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a$.

   (b) $\exists x \in \mathbb{Q}$ so that x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a$.

   (c) $\forall x \in \mathbb{Q}$, x can be written as a/b where $a, b \in \mathbb{Z}$ and $b|a^2$.

   (d) $\forall x \in \mathbb{R}$, if x is irrational then x^2 is irrational or x^3 is irrational.

   (e) $\forall x \in \mathbb{R}$, if x^2 is rational then x^3 is rational or x^5 is rational.

   (a) This statement is **false**. The negation of the statement is

   $$\exists x \in \mathbb{Q} \text{ so that } x \neq a/b \text{ for any } a, b \in \mathbb{Z} \text{ such that } b|a.$$

   An example which proves the negation is $x = 1/2$, because if $x = a/b$ and $b|a$, it would mean that x is an integer, and $1/2$ is not an integer.

   (b) This statement is **true**. An example proving this statement is $x = 1$, which can be written as $1/1$ where $1|1$.

   (c) This statement is **true**. Here is a proof. Let $x \in \mathbb{Q}$. This means that $x = c/d$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Then $x = \frac{cd}{d^2}$ where $d^2 \neq 0$, and $(cd)^2 = c^2d^2$ where $c^2 \in \mathbb{Z}$, so $d^2|(cd)^2$. Thus we can put $a = cd$ and $b = d^2$ and we get $x = a/b$ where $a, b \in \mathbb{Z}$ and $b|a^2$. Done.

   (d) This statement is **true**. We prove it by proving the *contrapositive* instead. The contrapositive is

   $$\forall x \in \mathbb{R}, \text{ if } x^2 \text{ is rational and } x^3 \text{ is rational, then } x \text{ is rational.}$$

   Assume that $x \in \mathbb{R}$ is such that x^2 is rational and x^3 is rational. We want to prove that x is rational. There are two cases.

   Case 1. Suppose $x = 0$. Then of course x is rational, so we are done.

   Case 2. Now suppose that $x \neq 0$. Since x^2 is rational, $x^2 = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and also $a \neq 0$ (since $x \neq 0$). Since x^3 is rational, $x^3 = c/d$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Thus $x = x^3/x^2 = (c/d)/(a/b) = bc/ad$, where bc and ad are integers with $ad \neq 0$ (since a and d are each nonzero), so x is rational.

   (e) This statement is **false**. The negation is

   $$\exists x \in \mathbb{R} \text{ so that } x^2 \text{ is rational but } x^3 \text{ is irrational and } x^5 \text{ is irrational.}$$

   This statement is proved by the example $x = \sqrt{2}$. Then $x^2 = 2$ is rational, but $x^3 = 2\sqrt{2}$ and $x^5 = 4\sqrt{2}$ are both irrational. Both of these statements follow from Exercise 10 on page 178 of the text, or could be proved by contradiction. For example, to prove that $2\sqrt{2}$ is irrational, we assume that it is rational.This means that $2\sqrt{2} = a/b$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. But then $\sqrt{2} = a/(2b)$ where $2b \in \mathbb{Z}$ and $2b \neq 0$, so $\sqrt{2}$ is rational, which is impossible. Therefore $2\sqrt{2}$ must be irrational. Similarly $4\sqrt{2}$ is irrational.

2. (a) Disprove the following statement: $\forall x, y \in \mathbb{R}$, if $\lfloor xy \rfloor = 0$ then $\lfloor x \rfloor = 0$ or $\lfloor y \rfloor = 0$.
   (b) Write out the contrapositive of the statement in part (a). Is it true or false? Explain.
   (c) Write out the converse of the statement in part (a). Is it true or false? Explain.
   (d) Prove or disprove the following statement: $\forall x, y \in \mathbb{R}$, if $\lceil xy \rceil = 0$ then $\lceil x \rceil = 0$ or $\lceil y \rceil = 0$.
   (e) Prove or disprove the following statement: $\exists x, y \in \mathbb{R}$ such that $\lfloor xy \rfloor = 0$ and $\lfloor x \rfloor \lfloor y \rfloor = 271$.
   (f) Prove or disprove the following statement: $\exists x, y \in \mathbb{R}$ such that $\lceil xy \rceil = 0$ and $\lceil x \rceil \lceil y \rceil = 271$.

   (a) To disprove this statement, we need to find real numbers x and y so that $\lfloor xy \rfloor = 0$ but $\lfloor x \rfloor \neq 0$ and $\lfloor y \rfloor \neq 0$. An example is $x = -1/2$ and $y = -1$. Then $xy = 1/2$, so $\lfloor xy \rfloor = \lfloor 1/2 \rfloor = 0$ while $\lfloor x \rfloor = \lfloor -1/2 \rfloor = -1 \neq 0$ and $\lfloor y \rfloor = \lfloor -1 \rfloor = -1 \neq 0$.

   Note: if x and y are both positive numbers so that $\lfloor xy \rfloor = 0$, then $0 < xy < 1$, so either $0 < x < 1$ or $0 < y < 1$, so one of $\lfloor x \rfloor = 0$ or $\lfloor y \rfloor = 0$ will have to be true. So for a counterexample we have to look at negative x and y.

   (b) The contrapositive is

$$\forall x, y \in \mathbb{R}, \text{ if } \lfloor x \rfloor \neq 0 \text{ and } \lfloor y \rfloor \neq 0 \text{ then } \lfloor xy \rfloor \neq 0.$$

   The contrapositive is **false** because it is equivalent to the original statement which is false.

   (c) The converse is

$$\forall x, y \in \mathbb{R}, \text{ if } \lfloor x \rfloor = 0 \text{ or } \lfloor y \rfloor = 0 \text{ then } \lfloor xy \rfloor = 0.$$

   The converse is **false**. A counterexample is $x = 1/2$ and $y = 2$. Then $\lfloor x \rfloor = \lfloor 1/2 \rfloor = 0$, so the "if" part is true, but $xy = 1$, so $\lfloor xy \rfloor = \lfloor 1 \rfloor = 1 \neq 0$, so the "then" part is false.

   (d) This statement is **false**. A counterexample is $x = 1/2$ and $y = -1$. Then $xy = -1/2$, so $\lceil xy \rceil = \lceil -1/2 \rceil = 0$, but $\lceil x \rceil = \lceil 1/2 \rceil = 1 \neq 0$ and $\lceil y \rceil = \lceil -1 \rceil = -1 \neq 0$.

   (e) This statement is **true**. An example is $x = -271$ and $y = -1/272$. Then $xy = 271/272$, so $\lfloor xy \rfloor = \lfloor 271/272 \rfloor = 0$, and $\lfloor x \rfloor \lfloor y \rfloor = \lfloor -271 \rfloor \lfloor -1/272 \rfloor = (-271)(-1) = 271$.

   (f) This statement is **false**. We prove this by contradiction. Suppose that the statement were true. Let x and y be real numbers so that $\lceil xy \rceil = 0$ and $\lceil x \rceil \lceil y \rceil = 271$. Since $\lceil xy \rceil = 0$, we must have $-1 < xy \leq 0$, so xy is negative (or zero). This means that (without loss of generality) $x \leq 0$ and $y \geq 0$. But then $\lceil x \rceil \leq 0$ and $\lceil y \rceil \geq 0$, and thus $\lceil x \rceil \lceil y \rceil \leq 0$, which contradicts $\lceil x \rceil \lceil y \rceil = 271$.

3. Let N be your student ID number.

   (a) **Use the Euclidean Algorithm** to find $\gcd(N, 271)$.
   (b) Use your answer to part (a) to write $\gcd(N, 271)$ in the form $Na + 271b$ where $a, b \in \mathbb{Z}$.
   (c) Suppose that M is a positive integer such that $\gcd(M, 271) = \gcd(M, 2008)$. Find $\gcd(M, 271)$. Explain. [*Hint*: 271 is prime.]

(d) Suppose that M is an integer between 250000 and 450000 such that $\gcd(M, 271) = \gcd(M, 2008) + 20$. Find M. Explain. [You may use Exercise 33, page 631.]

(a) Let's do it for the hypothetical student number $N = 123456$. The Euclidean algorithm gives:

$$
\begin{aligned}
123456 &= 455 \cdot 271 + 151 && (\text{so } 151 = 123456 - 455 \cdot 271) \\
271 &= 1 \cdot 151 + 120 && (\text{so } 120 = 271 - 151) \\
151 &= 1 \cdot 120 + 31 && (\text{so } 31 = 151 - 120) \\
120 &= 3 \cdot 31 + 27 && (\text{so } 27 = 120 - 3 \cdot 31) \\
31 &= 1 \cdot 27 + 4 && (\text{so } 4 = 31 - 27) \\
27 &= 6 \cdot 4 + 3 && (\text{so } 3 = 27 - 6 \cdot 4) \\
4 &= 1 \cdot 3 + 1 && (\text{so } 1 = 4 - 3) \\
3 &= 3 \cdot 1,
\end{aligned}
$$

so $\gcd(123456, 271) = \mathbf{1}$, the last nonzero remainder.

(b) Now, starting with the second-last equation above, solving it for the gcd 1, and plugging in the remainders one by one from the earlier equations, we get:

$$
\begin{aligned}
1 &= 4 - 3 \\
&= 4 - (27 - 6 \cdot 4) = 7 \cdot 4 - 27 \\
&= 7 \cdot (31 - 27) - 27 = 7 \cdot 31 - 8 \cdot 27 \\
&= 7 \cdot 31 - 8 \cdot (120 - 3 \cdot 31) = 7 \cdot 31 - 8 \cdot 120 + 24 \cdot 31 = 31 \cdot 31 - 8 \cdot 120 \\
&= 31 \cdot (151 - 120) - 8 \cdot 120 = 31 \cdot 151 - 39 \cdot 120 \\
&= 31 \cdot 151 - 39 \cdot (271 - 151) = 70 \cdot 151 - 39 \cdot 271 \\
&= 70 \cdot (123456 - 455 \cdot 271) - 39 \cdot 271 = 70 \cdot 123456 - 31850 \cdot 271 - 39 \cdot 271 \\
&\qquad = 70 \cdot 123456 - 31889 \cdot 271.
\end{aligned}
$$

So $a = 70$ and $b = -31889$ in this case.

(c) Since $\gcd(M, 271)$ is a divisor of 271 and 271 is prime, $\gcd(M, 271)$ must be either 1 or 271. Since 271 does not divide into 2008, $\gcd(M, 2008)$ cannot be 271. Thus, since $\gcd(M, 271) = \gcd(M, 2008)$, $\gcd(M, 271) = 1$.

(d) Since $\gcd(M, 271) = 1$ or 271 and $\gcd(M, 271) = \gcd(M, 2008) + 20 > 20$, $\gcd(M, 271)$ must equal 271 and thus $\gcd(M, 2008)$ must equal 251. Thus M must be a multiple of both 271 and 251, and so, since $\gcd(271, 251) = 1$ since 271 is prime, by Exercise 33, page 631, $271 \cdot 251 = 68021$ must divide into M. Thus $M = 68021k$ for some positive integer k. Moreover, since $\gcd(M, 2008) = 251$ is odd and 2008 is even, M cannot be even. So k must be odd. Now $68021 \cdot 3 = 204063$ is too small to be M and $68021 \cdot 7 = 476147$ is too large to be M, so M must be $68021 \cdot 5 = \mathbf{340105}$ which lies in the right range for M.

Note. By the way, $\gcd(M, 2008) = 251$ is possible because $251|2008$. In fact, $2008 = 251 \cdot 8$ so $\gcd(M, 2008) = \gcd(340105, 2008) = 251$ does hold.