MATH 271 ASSIGNMENT 2

Due 4:00 PM Friday, February 29, 2008. Hand your assignment (stapled please) to the marker (Jason Nicholson) in his office MS 390. (Or if nobody is in his office, put your assignment under the door.) Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. (a) Prove **by induction** (or by well-ordering) that $3^n + 4^n \leq 5^n$ for all integers $n \geq 2$.
   (b) Prove **by induction** (or by well-ordering) that $(5/4)^n - (3/4)^n \geq n/2$ for all integers $n \geq 1$. [*Note*: you might have to consider the cases $n = 1, 2, 3$ and $n \geq 4$ separately.]
   (c) Prove that, for all real numbers $x \geq 2$, if $(5/4)^x - (3/4)^x \geq x/2$ then $3^x + 4^x \leq 5^x$. Use this and part (b) to give another proof that $3^n + 4^n \leq 5^n$ for all integers $n \geq 2$.

2. The sequence $b_1, b_2, \ldots$ is defined by: $b_1 = 1$, and $b_n = \left\lceil \dfrac{n}{b_{n-1}} \right\rceil$ for all integers $n \geq 2$.
   (a) Find b_2, b_3, b_4, b_5 and b_6.
   (b) Use part (a) (and more data if you need it) to guess a simple formula for b_n in terms of n. [*Hint*: do the cases of odd n and even n separately.]
   (c) Use **induction** (or well-ordering) to prove your guess.
   (d) Suppose the sequence $c_1, c_2, \ldots$ is defined by: $c_1 = 1, c_2 = 1$, and $c_n = \left\lceil \dfrac{n}{c_{n-2}} \right\rceil$ for all integers $n \geq 3$. Calculate enough terms of the sequence to enable you to see a pattern. Use that pattern to guess what c_{271} and c_{281} are. (No proof needed — yet.)

3. You are given the following "while" loop:

   [*Pre-condition*: m is a nonnegative integer, $a = 0$, $b = 0$, $i = 0$.]

   while $(i \neq m)$
   1. $b := a + b + 1$
   2. $a := a - 4b$
   3. $i := i + 1$

   end while

   [*Post-condition*: $b = m(-1)^{m+1}$.]

   Loop invariant: $I(n)$ is

$$i = n, \quad a = \left\{ \begin{array}{ll} -2(n+1) & \text{if } n \text{ is odd} \\ 2n & \text{if } n \text{ is even} \end{array} \right\}, \quad b = n(-1)^{n+1}.$$

   (a) Prove the correctness of this loop with respect to the pre- and post-conditions.
   (b) Suppose the "while" loop is as above, except that statement 2 is replaced by: $a := a - b$. Run through the loop often enough, recording the various values of a and b that result, until you can predict what the post-condition value of b will be when $m = 271$. What is your prediction? Explain.