MATH 271 ASSIGNMENT 3 SOLUTIONS

1. For each integer $n \geq 1$, let $\mathcal{S}_n$ be the statement: for all sets $A, B_1, B_2, \ldots, B_n$,

$$(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_n) = A - (B_1 \cap B_2 \cap \cdots \cap B_n).$$

   (a) Prove that for all sets A, B and C, $(A - B) \cup (A - C) = A - (B \cap C)$. You may use the properties on page 272.

   (b) Prove **by induction** on n (or well ordering) that $\mathcal{S}_n$ is true for all integers $n \geq 1$.

   (a) Using the identities on page 272, we get

$$\begin{aligned}(A - B) \cup (A - C) &= (A \cap B^c) \cup (A \cap C^c) \quad \text{by \#12} \\ &= A \cap (B^c \cup C^c) \quad \text{by \#3(b) (distributive law)} \\ &= A \cap (B \cap C)^c \quad \text{by \#9(b) (De Morgan's Law)} \\ &= A - (B \cap C). \quad \text{by \#12}\end{aligned}$$

   (b) *Basis step.* When $n = 1$ the statement $\mathcal{S}_1$ says $A - B_1 = A - B_1$, which is obviously true.

   Inductive step. Assume that the statement $\mathcal{S}_k$ is true for all sets $A, B_1, B_2, \ldots, B_k$, where $k \geq 1$ is an integer. We want to prove that the statement $\mathcal{S}_{k+1}$ is true for all sets $A, B_1, B_2, \ldots, B_{k+1}$. So let $A, B_1, B_2, \ldots, B_{k+1}$ be sets. Then we want to prove that

$$(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_{k+1}) = A - (B_1 \cap B_2 \cap \cdots \cap B_{k+1}). \tag{1}$$

   Well,

$$\begin{aligned}&(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_{k+1}) \\ &\quad = \big((A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_k)\big) \cup (A - B_{k+1}) \\ &\quad = \big(A - (B_1 \cap B_2 \cap \cdots \cap B_k)\big) \cup (A - B_{k+1}) \quad \text{by assumption} \\ &\quad = A - (B_1 \cap B_2 \cap \cdots \cap B_k \cap B_{k+1}) \quad \text{by part (a)} \\ &\quad = A - (B_1 \cap B_2 \cap \cdots \cap B_{k+1})\end{aligned}$$

   which proves (1). This finishes the inductive step. Therefore by induction, $\mathcal{S}_n$ is true for all integers $n \geq 1$.

2. A sequence is called an *A-sequence* if every term is equal to 1, 2 or 3, and no two consecutive terms in the sequence are equal. Also, an A-sequence is called a *B-sequence* if the first and last terms are equal. So for example, 23121 is an A-sequence which is not a B-sequence, 23212 is both an A-sequence and a B-sequence, and 23122 is neither. For each integer $n \geq 2$, let a_n be the number of A-sequences of length n and let b_n be the number of B-sequences of length n.

   (a) Show that $a_n = 3 \cdot 2^{n-1}$ for all integers $n \geq 2$.

(b) Prove combinatorially that $b_n = a_{n-1} - b_{n-1}$ for all integers $n \geq 3$. [*Hint*: how can you make a B-sequence of length n from an A-sequence of length $n-1$?]

(c) Show $b_2 = 0$, and then use parts (a) and (b) to find b_3, b_4 and b_5.

(d) Use your answers to part (c) (and more if you need them) to guess a simple formula for b_n. [*Hint*: how far away is b_n from a nearby power of 2?]

(e) Use parts (a) and (b) to prove your formula in (d) **by induction** (or well ordering) for all integers $n \geq 2$.

(a) We count how many ways there are to make an A-sequence of length n. There are 3 choices for the first term (1, 2 or 3). After that each term can be any of 1, 2 or 3 except for what the previous term is, so there are 2 choices for each of the $n-1$ terms after the first. By the multiplication rule, the number of A-sequences of length n must be $a_n = 3 \cdot 2 \cdot 2 \cdots 2 = 3 \cdot 2^{n-1}$.

(b) To make a B-sequence of length n, we could take an A-sequence of length $n-1$ and add an nth term equal to the first term of the A-sequence. But for this to be allowed, the last two terms of the resulting sequence could not be equal, which means that the first and last terms of the original A-sequence cannot be equal. Thus we must start with an A-sequence which is **not** a B-sequence. There are a_{n-1} A-sequences of length $n-1$, and b_{n-1} of these are B-sequences, so there are $a_{n-1} - b_{n-1}$ A-sequences of length $n-1$ which are not B-sequences. Each of them can be converted to a B-sequence of length n as described above, and every B-sequence of length n will arise this way. Therefore the number b_n of B-sequences of length n must equal $a_{n-1} - b_{n-1}$.

(c) A B-sequence of length 2 must start and end with the same symbol, so both of its terms must be equal, but this is not allowed. So there are no B-sequences of length 2, that is, $b_2 = 0$. Now from parts (a) and (b) we get

- $b_3 = a_2 - b_2 = 3 \cdot 2^1 - 0 = 6$,
- $b_4 = a_3 - b_3 = 3 \cdot 2^2 - 6 = 12 - 6 = 6$,
- $b_5 = a_4 - b_4 = 3 \cdot 2^3 - 6 = 24 - 6 = 18$.

(d) Since

- $b_2 = 0 = 2 - 2 = 2^1 - 2$,
- $b_3 = 6 = 4 + 2 = 2^2 + 2$,
- $b_4 = 6 = 8 - 2 = 2^3 - 2$, and
- $b_5 = 18 = 16 + 2 = 2^4 + 2$,

we can guess that $b_n = 2^{n-1} - 2$ for n even and $b_n = 2^{n-1} + 2$ for n odd, which could be written as:

$$b_n = 2^{n-1} - 2(-1)^n \quad \text{for all integers } n \geq 2.$$

(e) *Basis step.* We already know that the formula $b_n = 2^{n-1} - 2(-1)^n$ is correct for $n = 2$, since $b_2 = 0$.

Inductive step. Assume that $b_k = 2^{k-1} - 2(-1)^k$ is true for some integer $k \geq 2$. Then

$$\begin{aligned} b_{k+1} &= a_k - b_k \quad \text{by part (b)} \\ &= 3 \cdot 2^{k-1} - (2^{k-1} - 2(-1)^k) \quad \text{by part (a) and by assumption} \\ &= 2 \cdot 2^{k-1} + 2(-1)^k = 2^k - 2(-1)^{k+1}, \end{aligned}$$

which proves that the formula for b_n is true for $n = k + 1$.
Therefore by induction, the formula for b_n is true for all integers $n \geq 2$.

Bonus question. Such a nice answer cries out for a nicer proof! Can you find a combinatorial proof that $b_n = 2^{n-1} - 2(-1)^n$? Maybe something like the proof in part (a), or something involving all subsets of an $(n-1)$-element set? But it would have to be rather clever, to account for the ± 2 in the formula. If you think you can get somewhere with this problem or have some ideas, tell your professor or TA. [*Warning*: neither of the professors in the course knows how to do this problem!]

3. A *balanced* subset of integers is a subset that has the same number of even integers as odd integers. For example, the subset $\{1, 2, 4, 9\}$ is balanced, but $\{1, 2, 3\}$ is not. For each positive integer n, let b_n be the number of balanced subsets of $\{1, 2, \ldots, 2n\}$.

   (a) Prove combinatorially that for all positive integers n,
   $$b_n = \sum_{i=0}^{n} \binom{n}{i}^2 = 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + 1.$$

   (b) Use part (a) to calculate b_1, b_2 and b_3.

   (c) Use your answers to part (b) to guess a formula for b_n which is a single binomial coefficient involving n. [*Hint*: write out Pascal's Triangle (page 359) up to $n = 6$.]

   (d) Prove your formula in part (c) combinatorially for all integers $n \geq 1$. [*Hint*: what if you construct a balanced subset of $\{1, 2, \ldots, 2n\}$ by choosing which even integers to put in your subset and which odd integers to *leave out* of your subset? How many integers would you choose altogether?]

   (a) We count the balanced subsets of $\{1, 2, \ldots, 2n\}$ by counting how many balanced subsets there are of each size separately and adding these numbers together. There is only one way to choose a balanced subset of no elements, namely the empty set. There are $\binom{n}{1}$ ways to choose one even integer from $\{1, 2, \ldots, 2n\}$ and $\binom{n}{1}$ ways to choose one odd integer from $\{1, 2, \ldots, 2n\}$, so by the multiplication rule there are $\binom{n}{1}\binom{n}{1} = \binom{n}{1}^2$ ways to choose a balanced subset of size 2. In general, for each $k \in \{0, 1, \ldots, n\}$, there are $\binom{n}{k}$ ways to choose k even integers from $\{1, 2, \ldots, 2n\}$ and $\binom{n}{k}$ ways to choose k odd integers from $\{1, 2, \ldots, 2n\}$, so by the multiplication rule there are $\binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$ ways to choose a balanced subset of size $2k$. Therefore by the addition rule, there are
   $$\sum_{k=0}^{n} \binom{n}{k}^2 = 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + 1$$
   balanced subsets of $\{1, 2, \ldots, 2n\}$ altogether, so this must equal b_n.

(b) From part (a),

- $b_1 = \binom{1}{0}^2 + \binom{1}{1}^2 = 1 + 1 = 2;$
- $b_2 = \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 1 + 2^2 + 1 = 6;$
- $b_3 = \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 1 + 3^2 + 3^2 + 1 = 20.$

(c) We know that $b_1 = 2 = \binom{2}{1}$. Looking at the fourth row $(1, 4, 6, 4, 1)$ of Pascal's Triangle we see that $\binom{4}{2} = 6 = b_2$. Looking at the sixth row $(1, 6, 15, 20, 15, 6, 1)$ of Pascal's Triangle we see that $\binom{6}{3} = 20 = b_3$. So we guess that $b_n = \binom{2n}{n}$ for every positive integer n.

(d) We can make a balanced subset of $\{1, 2, \ldots, 2n\}$ by choosing k even numbers from this set, and then choosing $n - k$ odd numbers to leave out (so that the remaining k odd numbers from $\{1, 2, \ldots, 2n\}$ are added in with the k even numbers to make a balanced set). Thus we are choosing $k + (n - k) = n$ numbers from $\{1, 2, \ldots, 2n\}$ to create the balanced set this way. Moreover, if we choose any set S of n numbers from $\{1, 2, \ldots, 2n\}$, we can make a balanced subset of $\{1, 2, \ldots, 2n\}$ out of S in the above way: if S includes (say) k even integers, then it must have $n - k$ odd integers, so there must be exactly k odd integers from $\{1, 2, \ldots, 2n\}$ which are *not* included in S; so just keep all the even integers in S plus all the odd integers in $\{1, 2, \ldots, 2n\}$ which are not in S, and we get a balanced subset of $\{1, 2, \ldots, 2n\}$. Thus the number of balanced subsets of $\{1, 2, \ldots, 2n\}$ is the same as the number of n-element subsets of $\{1, 2, \ldots, 2n\}$, which is $\binom{2n}{n}$. Thus $b_n = \binom{2n}{n}$.

Note. This fact is in Exercise 19, page 362.