MATH 271 ASSIGNMENT 3

Due 4:00 PM Thursday, March 20, 2008. Hand your assignment (stapled please) to the marker (Jason Nicholson) in his office MS 390. (Or if nobody is in his office, put your assignment under the door.) Assignments must be understandable to the marker (i.e., logically correct as well as legible), and of course must be done by the student in her/his own words. Answer **all** questions; but only one question per assignment will be marked for credit.

Marked assignments will be handed back during your scheduled lab, or in class.

1. For each integer $n \geq 1$, let $\mathcal{S}_n$ be the statement: for all sets $A, B_1, B_2, \ldots, B_n$,
$$(A - B_1) \cup (A - B_2) \cup \cdots \cup (A - B_n) = A - (B_1 \cap B_2 \cap \cdots \cap B_n).$$
   (a) Prove that for all sets A, B and C, $(A - B) \cup (A - C) = A - (B \cap C)$. You may use the properties on page 272.
   (b) Prove **by induction** on n (or well ordering) that $\mathcal{S}_n$ is true for all integers $n \geq 1$.

2. A sequence is called an *A-sequence* if every term is equal to 1, 2 or 3, and no two consecutive terms in the sequence are equal. Also, an A-sequence is called a *B-sequence* if the first and last terms are equal. So for example, 23121 is an A-sequence which is not a B-sequence, 23212 is both an A-sequence and a B-sequence, and 23122 is neither. For each integer $n \geq 2$, let a_n be the number of A-sequences of length n and let b_n be the number of B-sequences of length n.
   (a) Show that $a_n = 3 \cdot 2^{n-1}$ for all integers $n \geq 2$.
   (b) Prove combinatorially that $b_n = a_{n-1} - b_{n-1}$ for all integers $n \geq 3$. [*Hint*: how can you make a B-sequence of length n from an A-sequence of length $n - 1$?]
   (c) Show $b_2 = 0$, and then use parts (a) and (b) to find b_3, b_4 and b_5.
   (d) Use your answers to part (c) (and more if you need them) to guess a simple formula for b_n. [*Hint*: how far away is b_n from a nearby power of 2?]
   (e) Use parts (a) and (b) to prove your formula in (d) **by induction** (or well ordering) for all integers $n \geq 2$.

3. A *balanced* subset of integers is a subset that has the same number of even integers as odd integers. For example, the subset $\{1, 2, 4, 9\}$ is balanced, but $\{1, 2, 3\}$ is not. For each positive integer n, let b_n be the number of balanced subsets of $\{1, 2, \ldots, 2n\}$.
   (a) Prove combinatorially that for all positive integers n,
$$b_n = \sum_{i=0}^{n} \binom{n}{i}^2 = 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + 1.$$
   (b) Use part (a) to calculate b_1, b_2 and b_3.
   (c) Use your answers to part (b) to guess a formula for b_n which is a single binomial coefficient involving n. [*Hint*: write out Pascal's Triangle (page 359) up to $n = 6$.]
   (d) Prove your formula in part (c) combinatorially for all integers $n \geq 1$. [*Hint*: what if you construct a balanced subset of $\{1, 2, \ldots, 2n\}$ by choosing which even integers to put in your subset and which odd integers to *leave out* of your subset? How many integers would you choose altogether?]