MATHEMATICS 271 WINTER 2008
MDTERM SOLUTION Thursday, March 13, 2008.

[6] **1**. Use the Euclidean algorithm to find $\gcd(72, 17)$. Then use your work to write $\gcd(72, 17)$ in the form $72a + 17b$ where a and b are integers.
Solution: We have

$$\begin{aligned} 72 &= 4 \times 17 + 4 \\ 17 &= 4 \times 4 + 1 \\ 4 &= 4 \times 1 + 0, \end{aligned}$$

and so we get $\gcd(72, 17) = 1$, and
$\gcd(72, 17) = 1 = 17 - 4 \times 4 = 17 - 4 \times (72 - 4 \times 17) = 72 \times (-4) + 17 \times 17.$

Another way is using the "table method" as follows.

		72	17
	72	1	0
	17	0	1
$R_1 - 4R_2$	4	1	-4
$R_2 - 4R_3$	1	-4	17

Thus, $\gcd(72, 17) = 1$ and $\gcd(72, 17) = 72 \times (-4) + 17 \times 17$.

[6] **2**. Let $\mathcal{S}$ be the statement:

"For all sets A and B, if $A \cup B = \{1, 2\}$ then $\{1, 2\} \in \mathcal{P}(A) \cup \mathcal{P}(B)$."

(Here $\mathcal{P}(X)$ denotes the power set of the set X.)
(a) Write (as simply as possible) the *negation* of the statement $\mathcal{S}$.
Solution: The negation of the statement $\mathcal{S}$ is: "There exists sets A and B so that $A \cup B = \{1, 2\}$, but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$."

Comment: The negation of the statement $\mathcal{S}$ is NOT: "There exists sets A and B so that IF $A \cup B = \{1, 2\}$ THEN $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.". We see that both this statement and $\mathcal{S}$ are (vacuously) true when $A = B = \emptyset$.

(b) *Disprove* the statement $\mathcal{S}$.
Solution: We disprove the statement $\mathcal{S}$ by proving its negation stated above. The statement $\mathcal{S}$ is false because for example, when $A = \{1\}$ and $B = \{2\}$, $A \cup B = \{1, 2\}$, but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ because $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$.

[11] **3**. Let $\mathcal{S}$ be the statement: "For all integers n, if $6 \mid n$ then $9 \mid (n^2+3n)$."
(a) Is $\mathcal{S}$ true? Give a proof or disproof.
Solution: $\mathcal{S}$ is true and here is a proof. Let n be an integer and suppose that $6 \mid n$. Since $6 \mid n$, there exists an integer k such that $n = 6k$. Then $n^2 + 3n = (6k)^2 + 3 \times (6k) = 36k^2 + 18k = 9\,(4k^2 + 2k)$ where $4k^2 + 2k$ is an integer, which implies that $9 \mid (n^2 + 3n)$.

(b) Write (as simply as possible) the *contrapositive* of the statement $\mathcal{S}$. Is it true or false? Explain.
Solution: The contrapositive of the statement $\mathcal{S}$ is: "For all integers n, if $9 \nmid (n^2+3n)$ then $6 \nmid n$.". The contrapositive of the statement $\mathcal{S}$ is true because it is logically equivalent to $\mathcal{S}$ which is true as proven in part (a).

(c) Write (as simply as possible) the *converse* of the statement $\mathcal{S}$. Is it true or false? Explain.
Solution: The converse of the statement $\mathcal{S}$ is: "For all integers n, if $9 \mid (n^2+3n)$ then $6 \mid n$.". The converse of the statement $\mathcal{S}$ is false. For example, considering the integer $n = 3$, we see that $9 \mid (3^2 + 3 \times 3)$ because $3^2 + 3 \times 3 = 18 = 9 \times 2$, but $6 \nmid 3$.

[6] **4**. Of the following two statements, one is true and one is false. Prove the true statement. Disprove the false statement by writing out its negation and prove that. ($\mathbb{Z}$ denotes the set of all integers.)
(a) $\forall A \subseteq \mathbb{Z}\ \exists B \subseteq \mathbb{Z}$ so that $1 \in B - A$.
Solution: This statement is false. Its negation is: "$\exists A \subseteq \mathbb{Z}$ so that $\forall B \subseteq \mathbb{Z}$, $1 \notin B - A$.", and a proof of the negation is as follows. Let $A = \{1\}$. Then $A \subseteq \mathbb{Z}$, and for any set $B \subseteq \mathbb{Z}$, we see that $1 \notin B - A$ (because $1 \in A$). Actually, we can choose A to be any subset of $\mathbb{Z}$ which contains the element 1.

Comment: To prove the existence of such a set A, you must choose a specific subset A of $\mathbb{Z}$ first and then prove that for all subsets B of $\mathbb{Z}$, $1 \notin B - A$.

(b) $\forall A \subseteq \mathbb{Z}\ \exists B \subseteq \mathbb{Z}$ so that $1 \notin B - A$.
Solution: This statement is true and here is a proof. Let $A \subseteq \mathbb{Z}$. Let $B = \emptyset$. Then $B - A = \emptyset - A = \emptyset$, and $1 \notin \emptyset$ and so $1 \notin B - A$. Actually, we can choose B to be any subset of $\mathbb{Z}$ which does not contain the element 1.

Comment: Here, you must assume that A is an arbitrary subset of $\mathbb{Z}$ first and then choose a subset B of $\mathbb{Z}$ after (note that B can be expressed in term of A), and (2) you can not choose B to be the set $\{\emptyset\}$ which is not empty, and is not a subset of $\mathbb{Z}$.

[5] **5**. You are given that A and B are arbitrary subsets of the set $\mathbb{Z}$ of all integers such that $A \cap B = \{1\}$.
(a) Find an element of $A \times B$. Explain.
Solution: An element of $A \times B$ is $(1,1)$. Proof: Since $1 \in \{1\} = A \cap B$, we have $1 \in A$ and $1 \in B$, and so $1 \in A \cap B$.

(b) Find an element of the complement $(A \times B)^c$. (Here, assume the universal set is $\mathbb{Z} \times \mathbb{Z}$.) Explain.
Solution: An element of $(A \times B)^c$ is $(0,0)$. Proof (by contradiction): Suppose that $(0,0) \notin (A \times B)^c$. Then $(0,0) \in A \times B$, which means $0 \in A$ and $0 \in B$, and so $0 \in A \cap B$, which contradicts the assumption that $A \cap B = \{1\}$. Thus, $(0,0) \in (A \times B)^c$. Actually, we see that $(n,n) \in (A \times B)^c$ for any integer $n \neq 1$.

[6] **6**. Prove **using mathematical induction** (or well ordering) that $2^n + 2n \leq 3^n$ for all integers $n \geq 2$.
Solution: We prove that $2^n + 2n \leq 3^n$ for all integers $n \geq 2$ using **mathematical induction** on n.
Basis step: When $n = 2$, we have $2^n + 2n = 2^2 + 2 \times 2 = 8 \leq 9 = 3^2 = 3^n$. Thus, the statement is true for the case $n = 2$.

Inductive step: Let $k \geq 2$ be an integer and suppose that $2^k + 2k \leq 3^k$. We want to show that $2^{k+1} + 2(k+1) \leq 3^{k+1}$.

Now,

$$\begin{aligned}
2^{k+1} + 2(k+1) &= 2 \times 2^k + 2k + 2 \\
&< 2 \times 2^k + 2k + 2k && \text{because } 2 < 2k \text{ as } 1 < k \\
&= 2\left(2^k + 2k\right) \\
&\leq 2 \times 3^k && \text{by assumption} \\
&\leq 3 \times 3^k && \text{because } 2 \leq 3 \text{ and } 3^k > 0 \\
&= 3^{k+1}.
\end{aligned}$$

Thus, we prove the inductive step.

Therefore, we proved that $2^n + 2n \leq 3^n$ for all integers $n \geq 2$.

Comment: The most common mistake we found is that when students want to prove $2^{k+1} + 2(k+1) \leq 3^{k+1}$ (let's call this $\mathcal{P}$), they use $\mathcal{P}$ to derive another inequality $\mathcal{Q}$, then prove $\mathcal{Q}$ is true. This does not prove that $\mathcal{P}$ is true. However, you can prove $\mathcal{P}$ by first proving that $\mathcal{P}$ is equivalent to $\mathcal{Q}$ and then proving $\mathcal{Q}$.