THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271
FINAL EXAMINATION, WINTER 2005
TIME: 3 HOURS

NAME______________________________ID____________Section________

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10	
Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(100, 87)$. Then write the gcd in the form $100x + 87y$ where x and y are integers.

(b) Use part (a) to find an inverse for 87 modulo 100.

[12] 2. Let $\mathcal{S}$ be the statement "For all integers a and b, if $a|b$ then $(10a) \mid (5b)$."

(a) Is $\mathcal{S}$ true? Give a proof or disproof.

(b) Write out the *converse* of $\mathcal{S}$, and give a proof or disproof.

(c) Write out the *contrapositive* of $\mathcal{S}$, and give a proof or disproof.

3. Let X be the set $\{1, 2, \dots, 10\}$, and define the relation $\mathcal{R}$ on the power set $\mathcal{P}(X)$ of all subsets of X by: for any sets $A, B \in \mathcal{P}(X)$, $A\mathcal{R}B$ if and only if the intersection $A \cap B$ contains **at most** one element.

[6] (a) Is $\mathcal{R}$ reflexive? Symmetric? Transitive? Give reasons.

[4] (b) Suppose $A = \{1, 2, 3\}$. Find the number of sets B in $\mathcal{P}(X)$ so that $A\mathcal{R}B$.

4. **Z** denotes the set of all integers.

[2] (a) Define the statement $a \equiv b \pmod 9$, where $a, b \in \mathbf{Z}$.

[2] (b)We know that $\equiv \pmod 9$ is an equivalence relation on **Z**. Find three elements in the equivalence class [19].

[4] (c) Suppose that $a, b \in \mathbf{Z}$ satisfy $a \equiv b \pmod 9$. Prove by contradiction that $a + 1 \not\equiv b \pmod 9$, using the definition of $\equiv \pmod 9$.

[4] 5. (a) Give an example of an equivalence relation on the set $\{a, b, c, d\}$ that has exactly three equivalence classes.

(b) Find and simplify the **number** of equivalence relations on the set $\{a, b, c, d\}$ that have exactly three equivalence classes.

[8] 6. (a) **Z** denotes the set of all integers. Prove or disprove each of the following two statements:

(i) $\forall a \in \mathbf{Z}$, $2a$ is odd or $3a$ is even.

(ii) $\exists a \in \mathbf{Z}$ such that: if $2a$ is odd then $3a$ is even.

(b) Write the **negation** of each statement in part (a).

7. Let $A = \{1, 2, 3\}$. One of the following two statements is true and one is false. Prove or disprove each statement.

[3] (a) $\exists$ a function $f : A \to A$, so that $\forall g : A \to A$, $(f \circ g)(1) = 1$.

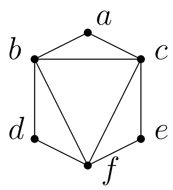
[3] (b) $\exists$ a function $f : A \to A$, so that $\forall g : A \to A$, $(g \circ f)(1) = 1$.

8. Let $A = \{1, 2, 3\}$, and let the function $f : A \to A$ be defined by: $f(1) = 2$, $f(2) = 1$, $f(3) = 1$.

[3] (a) Find the number of functions $g : A \to A$ so that $(f \circ g)(1) = 1$.

[3] (b) Find the number of functions $g : A \to A$ so that $(g \circ f)(1) = 1$.

9. Let G be the graph



[2] (a) Find an Euler circuit in G.

[2] (b) Find a Hamiltonian circuit in G.

[2] (c) Find a subgraph of G isomorphic to the complete graph K_3.

[4] (d) Draw two graphs with the following properties: each graph has exactly six vertices, of which four vertices have degree 2 and two vertices have degree 4; also one of the graphs contains a subgraph isomorphic to K_3, and the other graph does not contain a subgraph isomorphic to K_3.

[8] 10. Let the sequence (a_n) for $n \geq 1$ be defined by: $a_1 = 2$, and $a_n = 8n - 3a_{n-1} - 6$ for all integers $n \geq 2$. Prove by mathematical induction that $a_n = 2n$ for all $n \geq 1$.

$\overline{[80]}$