MATHEMATICS 271 WINTER 2006
MIDTERM SOLUTION

1. **Use the Euclidean algorithm** to find $\gcd(106, 20)$.
Solution: From

$$\begin{aligned} 106 &= 5 \times 20 + 6 \\ 20 &= 3 \times 6 + 2 \\ 6 &= 3 \times 2 + 0 \end{aligned}$$

and by Lemma 3.8.2, we have $\gcd(106, 20) = \gcd(20, 6) = \gcd(6, 2) = \gcd(2, 0) = 2$.

2. Let $\mathcal{S}$ be the following statement: "For all integers n, if $3n + 7$ is even then n is odd."
(a) Prove statement $\mathcal{S}$. Use contradiction or the contrapositive. You may assume that every integer is either even or odd but not both, but otherwise use only the definitions of even and odd.
Solution:

Let n be an integer such that $3n + 7$ is even. We prove that n is odd by contradiction. Suppose that n is not odd, that is, we suppose that n is even. Since n is even, there is an integer k such that $n = 2k$, and hence, $3n+7 = 3(2k)+7 = 2(3k + 3)+1$ (where $3k+3$ is an integer) which implies that $3n + 7$ is odd. This contradicts the assumption that $3n + 7$ is even. Thus, n is odd.

(b) Write out (as simply as possible) the *negation* of the statement $\mathcal{S}$.
Solution: The *negation* of the statement $\mathcal{S}$ is: "There exists an integer n so that $3n + 7$ is even but n is even."

3. Let $\mathcal{S}$ be the statement:

$$\text{for all sets } A, B, C, \text{ if } A \cap B \neq \emptyset \text{ and } A \cap C \neq \emptyset \text{ then } B \cap C \neq \emptyset.$$

(a) Is $\mathcal{S}$ true? Give a proof or disproof.
Solution: $\mathcal{S}$ is not true. For example, when $A = \{1, 2\}, B = \{1\}$ and $C = \{2\}$, we see that $A \cap B = \{1\} \neq \emptyset$ and $A \cap C = \{2\} \neq \emptyset$ but $B \cap C = \emptyset$.
(b) Write out (as simply as possible) the *converse* of the statement $\mathcal{S}$, and give a proof or disproof.

Solution: The *converse* of the statement $\mathcal{S}$ is: "For all sets A, B, C, if $B \cap C \neq \emptyset$ then $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$.".
The converse of $\mathcal{S}$ is false. For example, when $A = \emptyset$, and $B = C = \{1\}$, we have $B \cap C = \{1\} \neq \emptyset$, but $A \cap B = A \cap C = \emptyset$.
(c) Write out (as simply as possible) the *contrapositive* of the statement $\mathcal{S}$, and give a proof or disproof.
Solution: The *contrapositive* of the statement $\mathcal{S}$ is: " For all sets A, B, C, if $B \cap C = \emptyset$ then $A \cap B = \emptyset$ or $A \cap C = \emptyset$." The contrapositive of $\mathcal{S}$ is false because it is logically equivalent to $\mathcal{S}$ which is false as seen in (a).

4. Of the following two statements, one is true and one is false. Prove the true statement and disprove the false statement. ($\mathbb{Z}$ denotes the set of all integers.)
(a) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ so that $3 \mid (n+m)$.
Solution: This statement is true and here is a proof. Let $n \in \mathbb{Z}$. Choose $m = -n$. Then $m \in \mathbb{Z}$ and $n + m = 0 = 3 \times 0$, so $3 \mid (n+m)$.

(b) $\exists n \in \mathbb{Z}$ so that $\forall m \in \mathbb{Z}$, $3 \mid (n+m)$.
Solution: This statement is false and we prove the negation of the statement; that is, we prove that $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ so that $3 \nmid (n+m)$. Let $n \in \mathbb{Z}$. Choose $m = 1 - n$. Then $m \in \mathbb{Z}$ and $n + m = 1$, and since $3 \nmid 1$ see that $3 \nmid (n+m)$.

5. Of the following two statements, one is true and one is false. Prove the true statement and disprove the false statement.
(a) For all sets A and B, if $(1,2) \in A \times B$ and $(1,3) \in A \times B$ then $(2,3) \in A \times B$.
Solution: This statement is false because when $A = \{1\}$ and $B = \{2,3\}$, we have $A \times B = \{(1,2),(1,3)\}$, and therefore $(1,2) \in A \times B$ and $(1,3) \in A \times B$ but $(2,3) \notin A \times B$.
(b) For all sets A and B, if $(1,2) \in A \times B$ and $(2,3) \in A \times B$ then $(1,3) \in A \times B$.
Solution: This statement is true and here is a proof. Let A and B be sets so that $(1,2) \in A \times B$ and $(2,3) \in A \times B$. Since $(1,2) \in A \times B$, we get $1 \in A$, and since $(2,3) \in A \times B$, we get $3 \in B$. Then, from $1 \in A$ and $3 \in B$, we get $(1,3) \in A \times B$.

6. Prove using mathematical induction that $2n - 1 < 3^n$ for all integers $n \geq 1$.
Solution:
Basis $(n = 1)$: When $n = 1$, we have $2n - 1 = 1 < 3 = 3^1 = 3^n$.
Inductive Step: Let $k \geq 1$ be an integer and suppose that

$$2k - 1 < 3^k. \qquad [IH]$$

We want to prove that $2(k+1) - 1 < 3^{k+1}$.
Now,

$$\begin{aligned}
2(k+1) - 1 &= 2k - 1 + 2 \\
&< 3^k + 2 && \text{by } [IH] \\
&< 3^k + 3^k && \text{because } 2 < 3^k \text{ when } k \geq 1 \\
&< 3^k + 3^k + 3^k && \text{because } 3^k > 0 \\
&= 3 \times 3^k \\
&= 3^{k+1}.
\end{aligned}$$

Hence, we proved that $2(k+1) - 1 < 3^{k+1}$, and by the Principle of Mathematical Induction we conclude that $2n - 1 < 3^n$ for all integers $n \geq 1$.