MATHEMATICS 271 MIDTERM March 8, 2007

[6] 1. **Use the Euclidean algorithm** to find $\gcd(100, 43)$. Then use your work to write $\gcd(100, 43)$ in the form $100a + 43b$ where a and b are integers.

We get

$$\begin{aligned} 100 &= 2 \cdot 43 + 14, \\ 43 &= 3 \cdot 14 + 1, \\ 14 &= 14 \cdot 1 + 0, \end{aligned}$$

so $\gcd(100, 43) = 1$. Therefore we get

$$1 = 43 - 3 \cdot 14 = 43 - 3(100 - 2 \cdot 43) = 43 - 3 \cdot 100 + 6 \cdot 43 = 7 \cdot 43 - 3 \cdot 100.$$

Alternatively we could use the "table method", getting

		100	43
	100	1	0
	43	0	1
$R1 - 2R2$	14	1	-2
$R2 - 3R3$	1	-3	7

Therefore $\gcd(100, 43) = 1$ and $1 = -3(100) + 7(43)$.

[7] 2. Let $\mathcal{S}$ be the following statement:

for all reals n, if n^2 is irrational then n is irrational.

(a) Prove statement $\mathcal{S}$. Use contradiction or the contrapositive. Use only the definitions of rational and irrational.

Using contradiction: Assume that n is an arbitrary real number so that n^2 is irrational. We want to prove that n is irrational. To get a contradiction, assume that n is rational. By definition this means that $n = a/b$ for some $a, b \in \mathbb{Z}$ $(b \neq 0)$. Then $n^2 = (a/b)^2 = a^2/b^2$, where a^2 and b^2 are integers since $a, b \in \mathbb{Z}$ (and $b^2 \neq 0$). Thus by definition n^2 is rational, which contradicts our assumption that n^2 is irrational. Therefore n must be irrational.

Using the contrapositive: The contrapositive is:

for all reals n, if n is rational then n^2 is rational.

So to prove this, assume that n is an arbitrary rational number. We want to prove that n^2 is rational. By definition our assumption means that $n = a/b$ for some $a, b \in \mathbb{Z}$ $(b \neq 0)$. Then $n^2 = (a/b)^2 = a^2/b^2$, where a^2 and b^2 are integers since $a, b \in \mathbb{Z}$ (and $b^2 \neq 0$). Thus by definition n^2 is rational.

(b) Write (as simply as possible) the *negation* of statement $\mathcal{S}$.

It is: there exists a real n such that n^2 is irrational and n is rational.

[10] 3. Of the following four statements, three are true and one is false. Prove the true statements and disprove the false statement. $\mathbb{Z}$ denotes the set of all integers.

(a) $\exists A \subseteq \mathbb{Z}$ so that $A - \{1\} = A - \{2\}$.

This statement is true. An example is $A = \emptyset$. Then $A - \{1\} = \emptyset$ and $A - \{2\} = \emptyset$, so $A - \{1\} = A - \{2\}$. [We could also use any set A not containing 1 or 2.]

(b) $\exists A \subseteq \mathbb{Z}$ so that $A \cup \{1\} = A \cup \{2\}$.

This statement is true. An example is $A = \{1, 2\}$. Then $A \cup \{1\} = \{1, 2\}$ and $A \cup \{2\} = \{1, 2\}$, so $A \cup \{1\} = A \cup \{2\}$. [We could also use any set A containing both 1 and 2.]

(c) $\forall A \subseteq \mathbb{Z}\ \exists B \subseteq \mathbb{Z}$ so that $A - \{1\} = B - \{2\}$.

This statement is false. A counterexample is $A = \{2\}$. Then $A - \{1\} = \{2\}$, while for any set B we would have $2 \notin B - \{2\}$, so there cannot exist a set B so that $A - \{1\}$ will equal $B - \{2\}$. [We could also use any set A containing 2.]

(d) $\forall A \subseteq \mathbb{Z}\ \exists B \subseteq \mathbb{Z}$ so that $A \cap \{1\} = B - \{2\}$.

This statement is true. We prove it using two cases. Let A be an arbitrary subset of $\mathbb{Z}$.

Case (i). If $1 \in A$ then let $B = \{1\}$. Then $A \cap \{1\} = \{1\}$ and $B - \{2\} = \{1\}$, so $A \cap \{1\} = B - \{2\}$. [We could also use $B = \{1, 2\}$.]

Case (ii): If $1 \notin A$ then let $B = \emptyset$. Then $A \cap \{1\} = \emptyset$ and $B - \{2\} = \emptyset$, so again $A \cap \{1\} = B - \{2\}$. [We could also use $B = \{2\}$.]

[11] 4. Let $\mathcal{S}$ be the statement:

for all integers a and b, if $2|a$ and $3|b$, then $6|(ab)$.

(a) Is $\mathcal{S}$ true? Give a proof or disproof.

Yes, $\mathcal{S}$ is true. Here is a proof. Let $a, b \in \mathbb{Z}$ be arbitrary so that $2|a$ and $3|b$. This means that $a = 2k$ and $b = 3\ell$ for some $k, \ell \in \mathbb{Z}$. So $ab = (2k)(3\ell) = 6k\ell$, where $k\ell$ is an integer since both k and ℓ are integers. Therefore $6|(ab)$.

(b) Write out (as simply as possible) the *contrapositive* of statement $\mathcal{S}$, and give a proof or disproof.

The contrapositive is:

for all integers a and b, if $6 \nmid (ab)$, then $2 \nmid a$ OR $3 \nmid b$.

The contrapositive is true, because it is equivalent to the original statement $\mathcal{S}$ which is true.

(c) Write out (as simply as possible) the *converse* of statement $\mathcal{S}$, and give a proof or disproof.

The converse is:

for all integers a and b, if $6|(ab)$, then $2|a$ and $3|b$.

The converse is false. A counterexample is $a = 6$, $b = 1$. Then $ab = 6$, so $6|(ab)$, but $3 \not| b$. Another counterexample is $a = 3$ and $b = 2$; then $ab = 6$ so $6|(ab)$, but neither $2|a$ nor $3|b$ is true.

[6] 5. The sequence $x_0, x_1, x_2, \ldots$ is defined by:

$$x_0 = 1, \text{ and } x_n = 2x_{n-1} - 3n \text{ for all integers } n \geq 1.$$

Prove **using mathematical induction** that $x_n = 6 + 3n - 5(2^n)$ for all integers $n \geq 0$.

Basis Step. When $n = 0$ the statement says $x_0 = 6 + 3 \cdot 0 - 5(2^0) = 6 + 0 - 5 = 1$, which is true.

Inductive Step. Assume that $x_k = 6 + 3k - 5(2^k)$ for some integer $k \geq 0$. We want to prove that $x_{k+1} = 6 + 3(k+1) - 5(2^{k+1})$. Well,

$$\begin{aligned} x_{k+1} &= 2x_k - 3(k+1) \quad \text{by the given recursion} \\ &= 2[6 + 3k - 5(2^k)] - 3k - 3 \quad \text{by assumption} \\ &= 12 + 6k - 5(2^{k+1}) - 3k - 3 \\ &= 9 + 3k - 5(2^{k+1}) \\ &= 6 + 3(k+1) - 5(2^{k+1}), \end{aligned}$$

so the Inductive Step is proved.

Therefore $x_n = 6 + 3n - 5(2^n)$ for all integers $n \geq 0$.

[40]