UNIVERSITY OF CALGARY

Faculty of Science
Department of Mathematics & Statistics

Homework #1 - MATH 271 - L01 & L02

SOLUTIONS

Follow instructions available in the Assignment Policy document!

Question 1 (10 points) For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.

a: ([2 marks]) $(\forall x, y \in \mathbb{R}^+)\ \lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$

b: ([2 marks]) $(\forall x, y \in \mathbb{R}^+)$ if $y \geq 1$ then $\lfloor \frac{x}{y} \rfloor = \lfloor \frac{\lfloor x \rfloor}{\lfloor y \rfloor} \rfloor$

c: ([3 marks]) $(\forall x, y \in \mathbb{R}^+)$ if $y \geq 1$ then $\left\lfloor \frac{x}{y} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{\lfloor y \rfloor} \right\rfloor$

d: ([3 marks]) $(\forall n \in \mathbb{Z})\ \lceil \frac{n^2}{4} \rceil = \lceil \frac{n^2+3}{4} \rceil$ if and only if n is odd.

Solution:

a: This statement is false. The negation of the statement is:

$$(\exists x, y \in \mathbb{R}^+)\ \lfloor xy \rfloor \neq \lfloor x \rfloor \lfloor y \rfloor$$

An example which proves the negation is $x = y = 3/2$, because in this case

$$\lfloor xy \rfloor = \lfloor 9/4 \rfloor = 2, \text{ and } \lfloor x \rfloor \lfloor y \rfloor = \lfloor 3/2 \rfloor \lfloor 3/2 \rfloor = 1.$$

b: This statement is false. The negation of the statement is:

$$(\exists x, y \in \mathbb{R}^+)\ y \geq 1 \text{ and } \lfloor \frac{x}{y} \rfloor \neq \left\lfloor \frac{x}{y} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{\lfloor y \rfloor} \right\rfloor$$

An example which proves the negation is $x = 5/2$ and $y = 3/2$, because in this case

$$\left\lfloor \frac{x}{y} \right\rfloor = \lfloor 5/3 \rfloor = 1, \text{ and } \left\lfloor \frac{\lfloor x \rfloor}{\lfloor y \rfloor} \right\rfloor = \left\lfloor \frac{2}{1} \right\rfloor = 2.$$

c: This statement is true. We prove it directly as follows:

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x - \lfloor x \rfloor < \frac{1}{n}$, we need to prove that $\lfloor nx \rfloor = n\lfloor x \rfloor$.

But by multiplying by n we have: $nx - n\lfloor x \rfloor < n \cdot \frac{1}{n} = 1$,
so $nx < n\lfloor x \rfloor + 1$.
We always have $n\lfloor x \rfloor \leq nx$, so together yields
$n\lfloor x \rfloor \leq nx < n\lfloor x \rfloor + 1$
and therefore $\lfloor nx \rfloor = n\lfloor x \rfloor$ by definition.

d: This statement is true. Let $n \in \mathbb{Z}$, we need to prove implications in both directions.

I. Assume first that n is odd. We need to show that $\lceil \frac{n^2}{4} \rceil = \lceil \frac{n^2+3}{4} \rceil$.
Since n is odd, $n = 2k+1$ for some $k \in \mathbb{Z}$.
Therefore $\frac{n^2+3}{4} = \frac{(2k+1)^2+3}{4} = \frac{4k^2+4k+4}{4} = k^2+k+1 \in \mathbb{Z}$,
and thus $\lceil \frac{n^2+3}{4} \rceil = k^2+k+1$.
On the other hand $\frac{n^2}{4} = \frac{4k^2+4k+1}{4} = k^2+k+\frac{1}{4}$.
Thus $k^2+k < \frac{n^2+3}{4} \leq k^2+k+1$,
meaning $\lceil \frac{n^2}{4} \rceil = k^2+k+1$ by definition.
Hence $\lceil \frac{n^2}{4} \rceil = \lceil \frac{n^2+3}{4} \rceil$ as desired.

II. Now assume that $\lceil \frac{n^2}{4} \rceil = \lceil \frac{n^2+3}{4} \rceil$, we need to prove that n is odd.
We do so by proving the contrapositive, that is assuming that n is even, we show that $\lceil \frac{n^2}{4} \rceil \neq \frac{n^2+3}{4}$.
Since n is even, $n = 2k$ for some $k \in \mathbb{Z}$.
Therefore $\frac{n^2+3}{4} = \frac{(2k)^2+3}{4} = \frac{4k^2+3}{4} = k^2+\frac{3}{4}$.
We get $k^2 < \frac{n^2+3}{4} \leq k^2+1$,
and thus $\lceil \frac{n^2+3}{4} \rceil = k^2+1$ by definition.
On the other hand $\frac{n^2}{4} = \frac{4k^2}{4} = k^2 \in \mathbb{Z}$.
Thus $\lceil \frac{n^2}{4} \rceil = k^2$ by definition.
Hence $\lceil \frac{n^2}{4} \rceil \neq \lceil \frac{n^2+3}{4} \rceil$ as desired.

Question 2 For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.

a: The product of irrational numbers is irrational.

b: $\sqrt{6}$ is irrational.

Solution:

a: This statement is false. The negation of the statement is:

There are two irrational numbers whose product is rational.

Two such numbers are $\sqrt{2}$ and $\sqrt{2}$. Indeed by Theorem 3.7.1, $\sqrt{2}$ is irrational, however the product

$$\sqrt{2}\sqrt{2} = 2 \text{ is rational.}$$

b: The statement is true, and we prove it by contradiction supposing the statement is false.

That means the negation of the statement is true, that $\sqrt{6}$ is rational. There then are integers m and n with no common factors such that

$$\sqrt{6} = \frac{m}{n}$$

Squaring both sides we obtain

$$6 = \frac{m^2}{n^2},$$

or equivalently

$$m^2 = 6n^2. \tag{1}$$

This means that m^2 is divisible by 6, and we therefore claim that m itself is divisible by 6.
Indeed otherwise, by the Quotient Remainder Theorem, m must have the form $6k+i$ for some integer $0 < i < 6$. But then $m^2 = (6k+i)^2 = 36k^2 + 12ki + i^2 = 6(6k^2 + 2ki) + i^2$. However i^2 is not divisible by 6 for any integer $0 < i < 6$, so m^2 would not be divisible by 6.

We must thus have $m = 6k$ for some $k \in \mathbb{Z}$. Replacing in equation 1, we get

$$(6k)^2 = 6n^2$$

and by simplifying

$$6k^2 = n^2. \tag{2}$$

As above, n^2 must be divisible by 6 and again we conclude that n it self is divisible by 6. But this now means that 6 is a a common factor to both m and n, contrary to our assumptions.

Question 3 For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.

a: $(\forall a, b \in \mathbb{Z})(a \neq 0 \vee b \neq 0)$ implies $gcd(a, b) = gcd(a, a - b)$.

b: $(\forall a, b \in \mathbb{Z})(a \neq 0 \vee b \neq 0)$ implies $gcd(a, b) = gcd(a + b, a - b)$.

Solution:

a: This statement is true. For a proof, let $a, b \in \mathbb{Z}$, not both zero. We show that $gcd(a, b) \leq gcd(a, a-b)$ and $gcd(a, a-b) \leq gcd(a, b)$.

I. $gcd(a, b) \leq gcd(a, a-b)$.
If d is any divisor if a and b, then $a = dk$ and $b = d\ell$ for some $k, \ell \in \mathbb{Z}$.
But by substitution $a-b = dk - d\ell = d(k-\ell)$ showing that d divides both a and $a-b$.
Since $gcd(a, a-b)$ is the largest divisor of a and $a-b$, we must have $d \leq gcd(a, a-b)$.
Finally since $gcd(a, b)$ is such a divisor of a and b, we have $gcd(a, b) \leq gcd(a, a-b)$.

II. $gcd(a, a-b) \leq gcd(a, b)$.
If d is any divisor if a and $a-b$, then $a = dk$ and $a-b = d\ell$ for some $k, \ell \in \mathbb{Z}$.
But by substitution $b = a-(a-b) = dk - d\ell = d(k-\ell)$ showing that d divides both a and b.
Since $gcd(a, b)$ is the largest divisor of a and b, we must have $d \leq gcd(a, b)$.
Finally since $gcd(a, a-b)$ is such a divisor of a and $a-b$, we have $gcd(a, a-b) \leq gcd(a, b)$.

b: This statement is false. The negation of the statement is:

$$(\exists a, b \in \mathbb{Z})(a \neq 0 \vee b \neq 0) \text{ and } gcd(a, b) \neq gcd(a+b, a-b).$$

Let $a = b = 1 \in \mathbb{Z}$. Then certainly $a \neq 0$ or $b \neq 0$ (in fact neither is), and $gcd(a, b) = 1$ while $gcd(a+b, a-b) = gcd(2, 0) = 2$.