Faculty of Science
Department of Mathematics & Statistics

Homework #2 - MATH 271 - L01 & L02

SOLUTIONS

Question 1 Write a detailed but pseudo-algorithm (in the style of the text) which on input of two sets A and B computes their intersection $A \cap B$.

Algorithm to compute $A \cap B$:
[Input sets A and B represented as one-dimensional arrays $a[1], a[2], ..., a[m]$ and $b[1], b[2], ..., b[n]$, respectively. Starting with $a[1]$ and for each successive $a[i]$ in A, a check is made to see whether $a[i]$ is in B. To do this, $a[i]$ is compared to successive elements of B. If $a[i]$ equals some element of B, then $a[i]$ is placed in an array C. If $a[i]$ is not equal to any element of B, then the algorithm moves to the next element of A. Th algorithm outputs the array $C = A \cap B$.]
Input: m [a positive integer], $a[1], a[2], ..., a[m]$ [a one dimensional array representing A].
n [a positive integer], $b[1], b[2], ..., b[n]$ [a one dimensional array representing B].

Algorithm Body:

$i, k := 1$, C:=array[]
while $(i \leq m)$
  $j := 1$, found:="no"
  while ($j \leq n$ and found="no")
    if $a[i] = b[j]$ **then** $c[k] = a[i]$, $k++$, found := "yes"
    $j++$
  end while
  $i++$
end while

Output: C [an array]

Question 2 For each true statement below, give a proof. For each false statement below, write out its negation, then give a proof of the negation.
Assume all sets are subsets of a universal set U.

a: For all sets A, B and C, $A - (B - C) = (A - B) - C$.

b: For all sets A, B and C, $A \times (B - C) = (A \times B) - (A \times C)$.

c: For all sets A, B, C and D, $\overline{a}).(A - B) \times (C - D) = (A \times C) - (B \times D)$.

d: For all sets A, B and C, $(A - B) \cup (B - C) = (A \cup B) - (B \cap C)$.

Solution:

a: This statement is false. The negation of the statement is:

$$\text{There are sets } A,\ B \text{ and } C \text{ such that } A - (B - C) \neq (A - B) - C.$$

Indeed let $A = B = C = \{1\}$. Then $B - C = \emptyset$, so $A - (B - C) = A = \{1\}$. However $A - B = \emptyset$, so $(A - B) - C = \emptyset$ as well.

Therefore $A - (B - C) \neq (A - B) - C$ as desired.

b: The statement is true. Consider sets A, B and C. We prove the statement by showing that the two sets on each side of the equality are contained in each other.

1: $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

If $(x, y) \in A \times (B - C)$, then $x \in A$ and $y \in B - C$ by definition. Thus $y \in B$ and so $(x, y) \in (A \times B)$. Since $y \notin C$, then $(x, y) \notin (A \times C)$. This means $(x, y) \in (A \times B) - (A \times C)$ as desired.

2: $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

If $(x, y) \in (A \times B) - (A \times C)$, then by definition $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. But since $x \in A$, then it must be that $y \notin C$ as otherwise $(x, y) \in A \times C$. Thus $y \in B - C$ and $(x, y) \in A \times (B - C)$.

Since both sets have the same elements, they are equal and this completes the proof.

c: This statement is false. The negation of the statement is:

$$\text{There are sets } A,\ B,\ C \text{ and } D \text{ such that } (A - B) \times (C - D) \neq (A \times C) - (B \times D).$$

Indeed let $A = C = D = \{1\}$ and $B = \emptyset$. Then $C - D = \emptyset$ so $(A - B) \times (C - D) = \emptyset$. On the other hand $(1, 1) \in (A \times C)$, and $B \times D = \emptyset$, so $(1, 1) \in (A \times C) - (B \times D)$.

This completes the proof.

d: The statement is true. Consider sets A, B and C. We prove the statement by showing each set on each side of the equality is contained in the other.

1: $(A - B) \cup (B - C) \subseteq (A \cup B) - (B \cap C)$.

Let $x \in (A - B) \cup (B - C)$, then either $x \in (A - B)$ or $x \in \cup(B - C)$.

If $x \in A - B$, then $x \in A$ and $x \notin B$, so in particular $x \in A \cup B$ and $x \notin B \cap C$. Thus $x \in (A \cup B) - (B \cap C)$.

Else then $x \in B - C$, then $x \in B$ and $x \notin C$, so in particular $x \in A \cup B$ and $x \notin B \cap C$. Thus $x \in (A \cup B) - (B \cap C)$.

2: $(A \cup B) - (B \cap C) \subseteq (A - B) \cup (B - C)$.

Let $x \in (A \cup B) - (B \cap C)$, then by definition $x \in (A \cup B)$ and $x \notin (B \cap C)$, and in particular $x \in A \cup B$.

Then either $x \in A$ or $x \in B$, from which either $x \in A - B$ or $x \in B$. If $x \in A - B$, then $x \in (A - B) \cup (B - C)$. If $x \in B$, then $x \notin C$ since $x \notin B \cap C$. Thus again $x \in (A - B) \cup (B - C)$.

This can also be proved using the identities of Theorem 5.2.2.

Question 3 Assume that B is a Boolean algebra with operations $+$ and $\cdot$.
For each true statement below, give a detailed proof. For each false statement below, write out its negation, then give a proof of the negation.
In your arguments, you can use any part of the definition of a Boolean algebra and the properties listed in Theorem 5.3.2.

a: $(\forall a, b \in B)(a + b = 1 \longleftrightarrow b \cdot \overline{a} = \overline{a})$.

b: $(\forall a, b \in B)(a \cdot b = 0 \longleftrightarrow b + \overline{a} = \overline{a})$.

c: $(\forall a, b, c \in B)(a + b = a + c \longrightarrow b = c)$.

Solution:

a: This statement is true. Consider a Boolean algebra B and elements $a, b \in B$. We prove the statement by proving the implication form left to right and vice-versa.

1: $a + b = 1 \longrightarrow b \cdot \overline{a} = \overline{a}$.

So suppose that $a + b = 1$. Then:

$$\begin{aligned}
\overline{a} &= \overline{a} \cdot 1 && \text{[by the Identity Law]} \\
& \overline{a} \cdot (a + b) && \text{[since } a + b = 1\text{]} \\
& \overline{a} \cdot a + \overline{a} \cdot b && \text{[by the Distributive Law]} \\
& a \cdot \overline{a} + b \cdot \overline{a} && \text{[by the Commutative Law]} \quad \text{Which is what we needed.} \\
& 0 + b \cdot \overline{a} && \text{[by the Complement Law]} \\
& b \cdot \overline{a} + 0 && \text{[by the Commutative Law]} \\
& b \cdot \overline{a} && \text{[by the Identity Law]}
\end{aligned}$$

2: $b \cdot \overline{a} = \overline{a}) \longrightarrow a + b = 1$.

So suppose that $b \cdot \overline{a} = \overline{a}$. Then:

$$\begin{aligned}
1 &= a + \overline{a} && \text{[by the Complement Law]} \\
& a + b \cdot \overline{a} && \text{[by assumption]} \\
& a + a \cdot b + b \cdot \overline{a} && \text{[} a = a + ab \text{ by the Absorption Law]} \\
& a + b \cdot a + b \cdot \overline{a} && \text{[by the Commutative Law]} \\
& a + b \cdot (a + \overline{a}) && \text{[by the Distributive Law]} \\
& a + b \cdot 1 && \text{[by the Complement Law]} \\
& a + b && \text{[by the Identity Law]}
\end{aligned}$$

b: This statement is true. It can be proved in a similar manner as the above.

However it follows immediately from a: by duality. Indeed consider a Boolean algebra B and elements $a, b \in B$.

$$
\begin{aligned}
& a \cdot b = 0 \longleftrightarrow \\
& \qquad \longleftrightarrow \overline{a \cdot b} = \overline{0} && \text{[by the Double Complement Law]} \\
& \qquad \longleftrightarrow \overline{a} + \overline{b} = 1 && \text{[by the De Morgan and Complement Laws]} \\
& \qquad \longleftrightarrow \overline{b} \cdot \overline{\overline{a}} = \overline{\overline{a}} && \text{[by part a:]} \\
& \qquad \longleftrightarrow \overline{b} \cdot a = a && \text{[by the Double Complement Law]} \\
& \qquad \longleftrightarrow \overline{\overline{b} \cdot a} = \overline{a} && \text{[by the Double Complement Law]} \\
& \qquad \longleftrightarrow \overline{\overline{b}} + \overline{a} = \overline{a} && \text{[by the De Morgan Law]} \\
& \qquad \longleftrightarrow b + \overline{a} = \overline{a} && \text{[by the Double Complement Law]}
\end{aligned}
$$

c: This statement is false in general. It turns out to be true in some Boolean algebra (for example B the power set of the empty set), but it does not follow from the axioms of Boolean algebras, that is it is not true for all Boolean algebras.

Indeed there is a Boolean algebra B and $a, b, c \in B$ such that $a + b = a + c$ but $b \neq c$.

To see this let B be the following powerset Boolean algebra $B = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ with the usual operations $+ = \cup$ and $\cdot = \cap$.

If $a = b = \{1\}$ and $c = \emptyset$, then $a + b = a + c = \{1\}$, but $b \neq c$.