Department of Mathematics and Statistics
University of Calgary

Math 311 Assignment

Due November 9th, 2007

Definition: Let A and B be nxn matrices. A is similar to B ($A \sim B$) if and only if there exists an nxn invertible matrix P so that $P^{-1} A P = B$.

Question:

1. If A, B, and C are nxn matrices. Show that
   - a. If A is similar to B, then B is similar to A.
   - b. A is similar to A.
   - c. If A is similar to B and B is similar to C, then this implies that A is similar to C.

Definition: An nxn matrix A is diagonalizable exactly when there exists an invertible nxn matrix P such that $P^{-1} A P$ is a diagonal matrix.
[Notice that an equivalent definition might be: An nxn matrix A is diagonalizable exactly when A is similar to a diagonal matrix.

Question:

2. Give an example of
   - (i) a 2x2 diagonalizable matrix.
   - (ii) a 3x3 diagonalizable matrix.
   - (iii) a 2x2 matrix which is not diagonalizable..

Definition: If A is an nxn matrix, a number λ is an eigenvalue of A if $A X = \lambda X$ for some non-zero column $X \in \mathbb{R}^n$.

Example: Consider the 2x2 matrix $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$.

If $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, *then* $AX = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Notice that $\lambda = -1$ is an eigenvalue for the matrix A and $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A which corresponds to the eigenvalue -1.

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Remark: Given an nxn matrix A, to find the eigenvalues, λ for A, we must have $AX = \lambda X \quad \textit{for some} \quad \mathbf{0} \neq X \in \mathbb{R}^n$.
This means that

$$AX - \lambda X = \mathbf{0}$$

$$\therefore \; AX - \lambda I_n X = \mathbf{0}$$

$$\therefore \; (A - \lambda I_n) X = \mathbf{0} \qquad (1)$$

Therefore we need to find λ so that the system of equations in (1) will have non-trivial solutions. This means that the matrix $A - \lambda I_n$ must not be invertible. Consequently we must have $\det (A - \lambda I_n) = 0$.

$\det (A - \lambda I_n)$ is the characteristic polynomial of the matrix A.

Example: If $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$ and λ is an eigenvalue of A, then,

$$A - \lambda I_2 = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix}$$

$$\therefore \; |A - \lambda I_2| = \det \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix}$$

$$= (1 - \lambda)(2 - \lambda) - 6$$

$$= \lambda^2 - 3\lambda + 2 - 6$$

$$= \lambda^2 - 3\lambda - 4.$$

Solving the equation $\det (A - \lambda I_2) = 0$, we get $\lambda = 4 \;\; or \;\; \lambda = -1$.
We conclude that the matrix A has two eigenvalues -1 and 4.

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Questions:

3. In each case determine the eigenvalues of the matrices:

(*i*) $A = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ -1 & 1 & 4 \end{pmatrix}$ (*ii*) $B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

(*iii*) $C = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ (*iv*) $D = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ -4 & 4 & 0 \end{pmatrix}$

Remark: Let A be an nxn matrix with an eigenvalue λ. Let $E_\lambda = \{X \in \mathbb{R}^n : AX = \lambda X\}$. Then every $X \in E_\lambda$, $X \neq \mathbf{0}$ is an eigenvector associated with the eigenvalue λ. [E_λ is called an eigenspace.]

Question:

4. Prove that E_λ as defined above is a subspace of $\mathbb{R}^n$..

5. Using the matrices given in question 3, determine the eigenvectors associated with each eigenvalue and describe the resulting eigenspaces. In each instance determine the dimension of the eigenspace.

6. If A and B are nxn matrices and if A is similar to B, show that
   a. det A = det B.
   b. tr A = tr B.
   c. A and B have the same characteristic polynomial.
   d. A and B have the same eigenvalues.
   e. If A is diagonalizable, then B is diagonalizable.