University of Calgary
Faculty of Science
Department of Mathematics and Statistics

Math 311 | Sheet 2 | Fall 2007

1. Determine whether or not the following sets with the indicated operations are vector spaces or not. Justify your answer.

   a. The set, V, of non-negative real numbers; with ordinary addition and scalar multiplication.

   b. The set, V, of all 2x2 matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$; with the usual operations of addition and scalar multiplication as for the vector space $M_{22}$.

   c. The set, V, of all 2x2 matrices which have determinant equal to zero; with the usual operations of addition and scalar multiplication as for the vector space $M_{22}$.

   d. The set V of all 2x2 matrices with the addition of $M_{22}$ but having scalar multiplication, *, defined by: $\textit{for} \quad a \in \mathbb{R}, \quad \textit{and} \quad X \in V, \quad a * X = aX^T$

   e. Let V be a set which consists of the ordered triples $(x, y, z)$ in $\mathbb{R}^3$ with addition defined as usual and with scalar multiplication defined as follows:
   $\textit{If} \quad a \in \mathbb{R}, \quad \textit{and} \quad (x, y, z) \in \mathbb{R}^3, \quad \textit{then} \quad a(x, y, z) = (ax, 0, az)$

   f. Let V be the set of all ordered pairs, $\begin{pmatrix} x \\ y \end{pmatrix}$, in $\mathbb{R}^2$ with scalar multiplication defined as usual and addition defined as follows:

   $$\text{If} \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2, \text{ then } \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \end{pmatrix}$$

2. Let V be the set of positive real numbers with vector addition being ordinary multiplication, and scalar multiplication being defined by $a\ \boldsymbol{v} = \boldsymbol{v}^a$. Show that V is a vector space.

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3. In each case show that the condition $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}$ in V implies that $a = b = c = 0$.

   a. $V = \mathbb{R}^4$; $\boldsymbol{u} = (2,1,0,2)$, $\boldsymbol{v} = (1,1,-1,0)$, $\boldsymbol{w} = (0,1,2,1)$

   b. $V = P_3$; $\boldsymbol{u} = x^3 + x$, $\boldsymbol{v} = x^2 + 1$, $\boldsymbol{w} = x^3 - x^2 + x + 1$

4. Let V be a vector space which contains the vectors $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ... $\boldsymbol{v}_n$. Let $a_1$, $a_2$, ... $a_n$ be real numbers. Use induction on n to prove that

   a. $a(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \dots \boldsymbol{v}_n) = a\boldsymbol{v}_1 + a\boldsymbol{v}_2 + \dots + a\boldsymbol{v}_n$

   b. $(a_1 + a_2 + \dots a_n)\boldsymbol{v} = a_1\boldsymbol{v} + a_2\boldsymbol{v} + \dots + a_n\boldsymbol{v}$

5. Let $V$ be a vector space. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $V$, and if $a$ and $b$ are scalars, show that

   a. If $a\boldsymbol{v} = b\boldsymbol{v}$ and $\boldsymbol{v} \neq \mathbf{0}$, then $a = b$

   b. If $a\boldsymbol{u} = a\boldsymbol{v}$ and $a \neq 0$, then $\boldsymbol{u} = \boldsymbol{v}$