

Department of Mathematics and Statistics
University of Calgary
Sheet 5

Math 311

1. U and V are vector spaces. $T : U \rightarrow V$ is a linear transformation, prove that $\text{Ker } T$ is a subspace of U and $\text{Im } T$ is a subspace of V .
2. $T : M_{22} \rightarrow \mathbb{R}^2$ is defined as follows: $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + d \\ 2b \end{pmatrix}$.
 - a. Show that T is a linear transformation.
 - b. Determine $\text{Ker } T$ and $\text{Im } T$.
 - c. Say whether or not T is one-to-one or onto.
3. U and V are vector spaces. $T : U \rightarrow V$ is a linear transformation. If $\{\mathbf{u}_i\}_{i=1}^n$ is a set of vectors in U so that $\{T(\mathbf{u}_i)\}_{i=1}^n$ is a linearly independent subset of V , show that $\{\mathbf{u}_i\}_{i=1}^n$ is a linearly independent subset of U .
4. Give an example of a linear transformation T between the vector spaces U and V so that if $\{\mathbf{u}_i\}_{i=1}^n$ are linearly independent in V then $\{T(\mathbf{u}_i)\}_{i=1}^n$ is not a linearly independent set. [In other words, the converse of question 3 does not hold.].
5. $T : U \rightarrow V$ is a linear transformation between the vector spaces U and V . Prove that $\text{Ker } T = \{\mathbf{0}_U\}$ if and only if T is one-to-one.
6. Give an example of a linear transformation which is one-to-one and not onto. [Do not choose the trivial examples.
7. Give an example of a linear transformation which is onto but not one-to-one. [Do not choose the trivial examples.
8. V is a vector space. $T : V \rightarrow V$ is a linear transformation with the property that $T(T(\mathbf{v})) = \mathbf{v}$, for each \mathbf{v} in V . Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly independent if and only if $T(\mathbf{v}) \neq \pm \mathbf{v}$.

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9. Given that U and V are vector spaces and that $T : U \rightarrow V$ is a linear transformation, show that if T is onto and if $U = \text{span} \left\{ \left\{ \mathbf{u}_i \right\}_{i=1}^n \right\}$, then
- $$W = \text{span} \left\{ \left\{ T(\mathbf{u}_i) \right\}_{i=1}^n \right\}.$$

10. Let V be a vector space with vectors $\left\{ \mathbf{v}_i \right\}_{i=1}^n$. Define a mapping $T : \mathbb{R}^n \rightarrow V$ so

$$\text{that } T \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^n a_i \mathbf{v}_i.$$

- Show that T is linear.
 - Show that T is 1-1 if and only if $\left\{ \mathbf{v}_i \right\}_{i=1}^n$ is linearly independent in V .
 - Show that T is onto if and only if $V = \text{span} \left\{ \left\{ \mathbf{v}_i \right\}_{i=1}^n \right\}$.
11. U , V and W are vector spaces. $T : U \rightarrow V$ and $S : V \rightarrow W$ are linear transformations. Define a mapping $ST : U \rightarrow W$ as follows:
- $$ST(\mathbf{u}) = S(T(\mathbf{u})).$$
- Prove that ST is a linear transformation.
 - Prove that if ST is 1-1 then T is 1-1.
 - Prove that if ST is onto then S is onto.
 - Prove that if S and T are both 1-1 then ST is 1-1.
 - Prove that if S and T are both onto then ST is also onto.

12. T is a mapping between \mathbb{R}^4 and \mathbb{R}^3 defined as follows:

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + 2y - z + w \\ 3x + y + 2w \\ x - 3y + 2w \end{pmatrix}.$$

- Determine the matrix A associated with this mapping and verify that T is a linear transformation.
- Determine the rank of the matrix A .
- Determine the kernel and image of the mapping T and their respective dimensions.