Department of Mathematics and Statistics
University of Calgary

Math 311 Sheet 6

1. Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be vectors in $\mathbb{R}^n$. Show that

   a. $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$

   b. $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$

   c. $(k\,\boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot (k\,\boldsymbol{y}) = k\,(\boldsymbol{x} \cdot \boldsymbol{y})$

2. Given that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{0}$ if and only if $\| \boldsymbol{x} + \boldsymbol{y} \| = \| \boldsymbol{x} - \boldsymbol{y} \|$.

3. Given that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that $(\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) = 0$ if and only if $\| \boldsymbol{x} \| = \| \boldsymbol{y} \|$.

4. Show that $\| \boldsymbol{x} + \boldsymbol{y} \|^2 = \| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{0}$.

5. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$, show that

   a. $\boldsymbol{x} \cdot \boldsymbol{y} = \frac{1}{4}\left(\| \boldsymbol{x} + \boldsymbol{y} \|^2 - \| \boldsymbol{x} - \boldsymbol{y} \|^2 \right)$

   b. $\| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2 = \frac{1}{2}\left(\| \boldsymbol{x} + \boldsymbol{y} \|^2 + \| \boldsymbol{x} - \boldsymbol{y} \|^2 \right)$

6. Find a and b so that the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ are orthogonal.

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix}$$

7. The set $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of the vectors in B. Determine an orthonormal basis for $\mathbb{R}^3$.

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8. Given the set $B = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^4$, determine a, b, c, and d so that the set B is an orthogonal set.

9. In each case either show that the statement is true or give an example showing that the statement is false.

a. A set in $\mathbb{R}^4$ is linearly independent if and only if the set is an orthogonal set.

b. If $\{ \boldsymbol{x}, \boldsymbol{y} \}$ is an orthogonal set in $\mathbb{R}^n$, then $\{ \boldsymbol{x}, \boldsymbol{x} + \boldsymbol{y} \}$ is also an orthogonal set.

c. If $\{ \boldsymbol{x}_i \}_{i=1}^{2}$ and $\{ \boldsymbol{y}_j \}_{j=1}^{3}$ are both orthogonal sets in $\mathbb{R}^n$, and if $\boldsymbol{x}_i \cdot \boldsymbol{y}_j = 0$ for all *i* and *j*, then $\{ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3 \}$ is an orthogonal set in $\mathbb{R}^n$.