Department of Mathematics and Statistics
University of Calgary

Math 311 Sheet 7

1. If U is a subspace of $\mathbb{R}^n$, $U^{\perp} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \cdot \boldsymbol{u} = 0 \quad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^n \right\}$. Prove that

   a. $U^{\perp}$ is a subspace of $\mathbb{R}^n$.

   b. $U \cap U^{\perp} = \{ \boldsymbol{0} \}$

2. Prove that every orthogonal set of vectors in $\mathbb{R}^n$ is linearly independent.

3. Let U be a subspace of $\mathbb{R}^n$. Let $\{ \boldsymbol{e}_i \}_{i=1}^{k}$ be an orthogonal basis for U. Given that $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows: if $\boldsymbol{x} \in \mathbb{R}^n$,

$$T(\boldsymbol{x}) = proj_U(\boldsymbol{x}) \text{ where } proj_U(\boldsymbol{x}) = \sum_{i=1}^{k} \left(\frac{\boldsymbol{x} \cdot \boldsymbol{e}_i}{\boldsymbol{e}_i \cdot \boldsymbol{e}_i} \boldsymbol{e}_i \right)$$

   a. Show that T is a linear transformation.
   b. Describe Ker T.

4. If $\boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{R}^n$, prove that $\left\| \frac{1}{\| \boldsymbol{x} \|} \boldsymbol{x} \right\| = 1$.

5. $\{ \boldsymbol{e}_i \}_{i=1}^{k}$ Is an orthogonal set of vectors in $\mathbb{R}^n$. Show that $\left\| \sum_{i=1}^{k} \boldsymbol{e}_i \right\|^2 = \sum_{i=1}^{k} \| \boldsymbol{e}_i \|^2$.

6. Let U be a subspace of $\mathbb{R}^n$. Let $\{ \boldsymbol{e}_i \}_{i=1}^{k}$ be an orthonormal basis for U. Then for any $\boldsymbol{x} \in U$, $\boldsymbol{x} = \sum_{i=1}^{k} (\boldsymbol{x} \cdot \boldsymbol{e}_i)$.

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7. Use the Gram-Schmidt algorithm to convert the given basis of the vector space V to (a) an orthogonal basis; (b) an orthonormal basis.

i. $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\}; \quad V = \mathbb{R}^3.$

ii. $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}; \quad V = span\ (B)$

8. If U is a subspace of $\mathbb{R}^n$, show that $U^{\perp\perp} = U^{\perp}$.

9. Problems from section 8.1 in your text.

10. Problems from section 8.2 in your text.