Department of Mathematics and Statistics
University of Calgary

Math 311 Assignment

1. Let U, V, and W be vector spaces with $T: U \rightarrow V$ and $S: V \rightarrow W$ linear transformations. Prove that if ST is an isomorphism, then T is one-to-one and S is onto.

Definition: Let A and B be nxn matrices. A is similar to B ($A \sim B$) if and only if there exists an nxn invertible matrix P so that $P^{-1} A P = B$.

Question:
2 If A, B, and C are nxn matrices. Show that
a. If A is similar to B, then B is similar to A.
b. A is similar to A.
c. If A is similar to B and B is similar to C, then this implies that A is similar to C.

Definition: An nxn matrix A is diagonalizable exactly when there exists an invertible nxn matrix P such that $P^{-1} A P$ is a diagonal matrix.
[Notice that an equivalent definition might be: An nxn matrix A is diagonalizable exactly when A is similar to a diagonal matrix.

Question:
3. Give an example of (i) a 2x2 diagonalizable matrix.
(ii) a 3x3 diagonalizable matrix.
(iii) a 2x2 matrix which is not diagonalizable..

Definition: If A is an nxn matrix, a number λ is an eigenvalue of A if $A X = \lambda X$ for some non-zero column $X \in \mathbb{R}^n$.

Example: Consider the 2x2 matrix $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$.

If $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, *then* $AX = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Notice that $\lambda = -1$ is an eigenvalue for the matrix A and $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A which corresponds to the eigenvalue -1.

Remark: Given an nxn matrix A, to find the eigenvalues, λ for A, we must have $AX = \lambda X \quad for \quad some \quad \mathbf{0} \neq X \in \mathbb{R}^n$.
This means that

$$AX - \lambda X = \mathbf{0}$$

$$\therefore \quad AX - \lambda I_n X = \mathbf{0}$$

$$\therefore \quad (A - \lambda I_n) X = \mathbf{0} \qquad (1)$$

Therefore we need to find λ so that the system of equations in (1) will have non-trivial solutions. This means that the matrix $A - \lambda I_n$ must not be invertible. Consequently we must have $\det (A - \lambda I_n) = 0$.

$\det (A - \lambda I_n)$ is the characteristic polynomial of the matrix A.

Example: If $A = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}$ and λ is an eigenvalue of A, then,

$$A - \lambda I_n = \begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix}$$

$$\therefore \quad A - \lambda I_n = \det \begin{pmatrix} 1 - \lambda & -1 \\ -6 & 2 - \lambda \end{pmatrix}$$

$$= (1 - \lambda)(2 - \lambda) - 6$$

$$= \lambda^2 - 3\lambda + 2 - 6$$

$$= \lambda^2 - 3\lambda - 4.$$

Solving the equation $\det (A - \lambda I_n) = 0$, we get $\lambda = 4 \quad or \quad \lambda = -1$.
We conclude that the matrix A has two eigenvalues -1 and 4.

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Questions:

4. In each case determine the eigenvalues of the matrices:

(*i*) $A = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ -1 & 1 & 4 \end{pmatrix}$ (*ii*) $B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

(*iii*) $C = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ (*iv*) $D = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 5 & 2 \\ +4 & 4 & 0 \end{pmatrix}$

Remark: Let A be an nxn matrix with an eigenvalue λ. Let $E_\lambda = \{X \in \mathbb{R}^n : AX = \lambda X\}$. Then every $X \in E_\lambda$, $X \neq \mathbf{0}$ is an eigenvector associated with the eigenvalue λ. [E_λ is called an eigenspace.]

Question:

5. Prove that E_λ as defined above is a subspace of $\mathbb{R}^n$..

6. Using the matrices given in question 3, determine the eigenvectors associated with each eigenvalue and describe the resulting eigenspaces. In each instance determine the dimension of the eigenspace.

7. If A and B are nxn matrices and if A is similar to B, show that
   a. det A = det B.
   b. tr A = tr B.
   c. A and B have the same characteristic polynomial.
   d. A and B have the same eigenvalues.
   e. If A is diagonalizable, then B is diagonalizable.

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Math 311 Sheet 6

1. Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be vectors in $\mathbb{R}^n$. Show that

   a. $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$

   b. $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$

   c. $(k\boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot (k\boldsymbol{y}) = k(\boldsymbol{x} \cdot \boldsymbol{y})$

2. Given that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{0}$ if and only if $\| \boldsymbol{x} + \boldsymbol{y} \| = \| \boldsymbol{x} - \boldsymbol{y} \|$.

3. Given that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, show that $(\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) = 0$ if and only if $\| \boldsymbol{x} \| = \| \boldsymbol{y} \|$.

4. Show that $\| \boldsymbol{x} + \boldsymbol{y} \|^2 = \| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{0}$.

5. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$, show that

   a. $\boldsymbol{x} \cdot \boldsymbol{y} = \frac{1}{4}\left(\| \boldsymbol{x} + \boldsymbol{y} \|^2 - \| \boldsymbol{x} - \boldsymbol{y} \|^2 \right)$

   b. $\| \boldsymbol{x} \|^2 + \| \boldsymbol{y} \|^2 = \frac{1}{2}\left(\| \boldsymbol{x} + \boldsymbol{y} \|^2 + \| \boldsymbol{x} - \boldsymbol{y} \|^2 \right)$

6. Find a and b so that the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$ are orthogonal.

   $$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix} 2 \\ b \\ 1 \end{pmatrix}$$

7. The set $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of the vectors in B. Determine an orthonormal basis for $\mathbb{R}^3$.

8. Given the set $B = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^4$, determine a, b, c, and d so that the set B is an orthogonal set.

9. In each case either show that the statement is true or give an example showing that the statement is false.

   a. A set in $\mathbb{R}^4$ is linearly independent if and only if the set is an orthogonal set.

   b. If $\{ \boldsymbol{x}, \boldsymbol{y} \}$ is an orthogonal set in $\mathbb{R}^n$, then $\{ \boldsymbol{x}, \boldsymbol{x} + \boldsymbol{y} \}$ is also an orthogonal set.

   c. If $\{ \boldsymbol{x}_i \}_{i=1}^{2}$ and $\{ \boldsymbol{y}_j \}_{j=1}^{3}$ are both orthogonal sets in $\mathbb{R}^n$, and if $\boldsymbol{x}_i \cdot \boldsymbol{y}_j = 0$ for all *i* and *j*, then $\{ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3 \}$ is an orthogonal set in $\mathbb{R}^n$.