Department of Mathematics and Statistics
University of Calgary

Math 311 Sheet 5

1. U and V are vector spaces. $T : U \to V$ is a linear transformation, prove that Ker T is a subspace of U and Im T is a subspace of V.

2. $T : M_{22} \to \mathbb{R}^2$ is defined as follows: $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + d \\ 2b \end{pmatrix}$.

   a. Show that T is a linear transformation.

   b. Determine Ker T and Im T.

   c. Say whether or not T is one-to-one or onto.

3. U and V are vector spaces. $T : U \to V$ is a linear transformation. If $\{ \boldsymbol{u}_i \}_{i=1}^{n}$ is a set of vectors in U so that $\{ T(\boldsymbol{u}_i) \}_{i=1}^{n}$ is a linearly independent subset of V, show that $\{ \boldsymbol{u}_i \}_{i=1}^{n}$ is a linearly independent subset of U.

4. Give an example of a linear transformation T between the vector spaces U and V so that if $\{ \boldsymbol{u}_i \}_{i=1}^{n}$ are linearly independent in V then $\{ T(\boldsymbol{u}_i) \}_{i=1}^{n}$ is not a linearly independent set. [In other words, the converse of question 3 does not hold.].

5. $T : U \to V$ is a linear transformation between the vector spaces U and V. Prove that $Ker\ T = \{ \mathbf{0}_U \}$ if and only if T is one-to-one.

6. Give an example of a linear transformation which is one-to-one and not onto. [Do not choose the trivial examples].

7. Give an example of a linear transformation which is onto but not one-to-one. [Do not choose the trivial examples].

8. V is a vector space. $T : V \to V$ is a linear transformation with the property that $T(T(\boldsymbol{v})) = \boldsymbol{v}$. for each $\boldsymbol{v}$ in V. Show that $\{\boldsymbol{v}, T(\boldsymbol{v})\}$ is linearly independent if and only if $T(\boldsymbol{v}) \neq \pm \boldsymbol{v}$.

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9. Given that U and V are vector spaces and that $T : U \to V$ Is a linear transformation, show that if T is onto and if $U = span\left\{\{\,\boldsymbol{u}_i\,\}_{i=1}^{n}\right\}$, then $W = span\left\{\{\,T(\boldsymbol{u}_i)\,\}_{i=1}^{n}\right\}$.

10. Let V be a vector space with vectors $\{\,\boldsymbol{v}_i\,\}_{i=1}^{n}$. Define a mapping $T : \mathbb{R}^n \to V$ so that $T\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i$.

   a. Show that T is linear.

   b. Show that T is 1-1 if and only if $\{\,\boldsymbol{v}_i\,\}_{i=1}^{n}$ is linearly independent in V.

   c. Show that T is onto if and only if $V = span\left\{\{\,\boldsymbol{v}_i\,\}_{i=1}^{n}\right\}$.

11. U V and W are vector spaces. $T : U \to V$ and $S : V \to W$ are linear transformations. Define a mapping $ST : U \to W$ as follows: $ST(\boldsymbol{u}) = S(T(\boldsymbol{u}))$.
   a. Prove that ST is a linear transformation.
   b. Prove that if ST is 1-1 then T is 1-1.
   c. Prove that if ST is onto then S is onto.
   d. Prove that if S and T are both 1-1 then ST is 1-1.
   e. Prove that if S and T are both onto then ST is also onto.

12. T is a mapping between $\mathbb{R}^4$ and $\mathbb{R}^3$ defined as follows:

$$T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + 2y - z + w \\ 3x + y + 2w \\ x - 3y + 2w \end{pmatrix}.$$

   a. Determine the matrix A associated with this mapping and verify that T is a linear transformation.
   b. Determine the rank of the matrix A.
   c. Determine the kernel and image of the mapping T and their respective dimensions.