Department of Mathematics and Statistics
University of Calgary

Math 311 Sheet 7

1. If U is a subspace of $\mathbb{R}^n$, $U^\perp = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \cdot \boldsymbol{u} = 0 \quad for \quad all \quad \boldsymbol{x} \in \mathbb{R}^n \right\}$. Prove that

   a. $U^\perp$ is a subspace of $\mathbb{R}^n$.

   b. $U \cap U^\perp = \{ \boldsymbol{0} \}$

2. Prove that every orthogonal set of vectors in $\mathbb{R}^n$ is linearly independent.

3. Let U be a subspace of $\mathbb{R}^n$. Let $\left\{ \boldsymbol{e}_i \right\}_{i=1}^k$ be an orthogonal basis for U. Given that $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows: if $\boldsymbol{x} \in \mathbb{R}^n$,

$$T(\boldsymbol{x}) = proj_U(\boldsymbol{x}) \text{ where } proj_U(\boldsymbol{x}) = \sum_{i=1}^k \left(\frac{\boldsymbol{x} \cdot \boldsymbol{e}_i}{\boldsymbol{e}_i \cdot \boldsymbol{e}_i} \boldsymbol{e}_i \right)$$

   a. Show that T is a linear transformation.

   b. Describe Ker T.

4. If $\boldsymbol{0} \neq \boldsymbol{x} \in \mathbb{R}^n$, prove that $\left\| \frac{1}{\| \boldsymbol{x} \|} \boldsymbol{x} \right\| = 1$.

5. $\left\{ \boldsymbol{e}_i \right\}_{i=1}^k$ Is an orthogonal set of vectors in $\mathbb{R}^n$. Show that $\left\| \sum_{i=1}^k \boldsymbol{e}_i \right\|^2 = \sum_{i=1}^k \| \boldsymbol{e}_i \|^2$.

6. Let U be a subspace of $\mathbb{R}^n$. Let $\left\{ \boldsymbol{e}_i \right\}_{i=1}^k$ be an orthonormal basis for U. Then for any $\boldsymbol{x} \in U$, $\boldsymbol{x} = \sum_{i=1}^k \left(\boldsymbol{x} \cdot \boldsymbol{e}_i \right)$.

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7. Use the Gram-Schmidt algorithm to convert the given basis of the vector space V to (a) an orthogonal basis; (b) an orthonormal basis.

i. $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\}; \quad V = \mathbb{R}^3.$

ii. $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}; \quad V = span\ (B)$

8. If U is a subspace of $\mathbb{R}^n$, show that $U^{\perp\perp} = U^{\perp}$.

9. Problems from section 8.1 in your text.

10. Problems from section 8.2 in your text.

11. Problems from section 8.3 in the text.

12. Problems from sections 9.1, and 9.2 in the text.