

Department of Mathematics, and Statistics
University of Calgary
Sheet 8

Math 311

1. Let S be a set of vectors in \mathfrak{R}^4 such that $S = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.
- Determine whether or not this set is linearly independent.
 - Let U be the subspace spanned by S . Determine the dimension of U and construct an orthogonal basis for U .
 - Extend the basis for U to an orthogonal basis for \mathfrak{R}^4 .
2. Say whether or not the following subsets of M_{22} are subspaces of M_{22} :
- $U = \{A \in M_{22}: AB = \bar{0}, B \text{ is a fixed matrix in } M_{22}\}$.
 - $U = \{A \in M_{22}: A \text{ is not invertible in } M_{22}\}$
 - $U = \{A \in M_{22}: BAC = CAB \text{ } B \text{ and } C \text{ are fixed matrices in } M_{22}\}$
3. Let U be a non-empty subset of a vector space V . Show that U is a subspace of V if and only if $u_1 + a u_2$ lies in U for all u_1 and u_2 in U and all $a \in \mathfrak{R}$.
4. If U and W are subspaces of a vector space V , let $Q = U \cup W = \{v \in V: v \in U \text{ or } v \in W\}$. Show that Q is a subspace of V if and only if either $U \subseteq W$ or $W \subseteq U$.
5. Let A be a square matrix with eigenvalue ν and corresponding eigenvector x . Show that if A is invertible, then the eigenvalue of its inverse, B , is $1/\nu$ and that x is the eigenvector of B associated with $1/\nu$.

Department of Mathematics, and Statistics
University of Calgary
Sheet 8

Math 311

6. Suppose that λ is an eigenvalue of a square matrix A with non-zero eigenvector X , show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of the matrix $A^3 - 2A + 3I$.
7. Assume that the 2×2 matrix A is similar to an upper triangular matrix. If $\text{tr } A = 0 = \text{tr } A^2$, show that $A^2 = 0$.
8. Let V be a vector space and let $S = \{e_i\}_{i=1}^n$ be a subset of V .
 - a. Define what is meant by the statement “ S is a linearly independent set of vectors in the vector space V ”.
 - b. Define what is meant by the statement “ S is a basis for the vector space V ”.
 - c. $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is a linearly independent subset of a vector space V . Say whether or not the following sets are linearly independent or linearly dependent. In each case, justify your answer.
 - (i) $A = \{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{z}, \mathbf{z}-\mathbf{u}\}$
 - (ii) $B = \{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{z}\}$
9. Define what is meant by each of the following statements:
 - a. V is a subspace of a vector space W .
 - b. A vector space V has dimension equal to n .
 - c. V and W are vector spaces. A linear transformation T from V to W is one-to-one.
 - d. V and W are vector spaces. A linear transformation T from V to W is onto.
 - e. Two vectors X and Y in \mathfrak{R}^n are orthogonal.
 - f. The matrix B has an eigenvalue λ and an eigenvector X .
10. Given that $S = \{e_i\}_{i=1}^n$ is an orthogonal set of n vectors in a vector space V , show that S is a linearly independent set of vectors.
11. Use the Gram Schmidt Orthogonalization lemma to transform the given set B into an

Department of Mathematics, and Statistics
University of Calgary
Sheet 8

Math 311

orthogonal basis for \mathfrak{R}^4 .

$$B = \{ (1,1,1,0), (0,1,1,1), (1,0,1,1), (1,1,0,1) \}.$$

12. Define what is meant by the statement “the $n \times n$ matrices A and B are similar”. Show that if A and B are similar $n \times n$ matrices, then the following statements are true.

a. $C_A(\lambda) = C_B(\lambda)$

b. $\det A = \det B$

c. $tr A = tr B$

13. Show that the vectors \mathbf{X} and \mathbf{Y} are orthogonal in \mathfrak{R}^n if and only if $\|\mathbf{X} + \mathbf{Y}\| = \|\mathbf{X} - \mathbf{Y}\|$.

14. Given that T is a linear transformation from a vector space V to a vector space W , show that

- $\text{Ker } T$ is a subspace of V .
- $\text{Im } T$ is a subspace of W .
- T is one-to-one if and only if $\text{Ker } T = \{0\}$.

15. T is a mapping from \mathfrak{R}^3 to \mathfrak{R}^4 defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + z \\ x - y \\ x - z \\ 3y - 3z \end{pmatrix}$$

- Show that T is a linear transformation.
- Determine the transformation matrix associated with T .
- Determine a basis for $\text{Ker } T$.
- Determine a basis for $\text{Im } T$.
- Determine the rank and nullity of T .

Department of Mathematics, and Statistics
University of Calgary
Sheet 8

Math 311

16. The set $B = \{(1,1,1), (1,0,1), (0,1,1)\}$ and the set $D = \{(1,0,2), (0,1,2), (1,2,0)\}$ are ordered bases for the vector space \mathbb{R}^3 . Determine the transition matrix from B to D and the transition matrix from D to B .
17. a. Define what is meant by “The matrix P is orthogonal”.
b. Say whether or not the matrices given below are orthogonal or not. If the matrix is orthogonal find the inverse of the matrix.

$$i. \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad ii. \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -2\sqrt{6} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

18. Show that the matrix A is diagonalizable and find the matrix P which diagonalizes A .

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$$

19. Find the bases for the row and column spaces of the matrix, A given by:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{pmatrix}$$

Determine the rank of A .

Department of Mathematics, and Statistics
University of Calgary
Sheet 8

Math 311

20. Consider the linear transformations:

$$T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3 ; \quad T_1 (x,y) = (x - y, x + 3y, 4y)$$

$$T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2 ; \quad T_2 (x,y,z) = (2x + y - 3z, x + 2y - z)$$

- a. Show that T_1 is one to one but not onto.
- b. Show that T_2 is onto but not one to one.
- c. Find a formula for $T_2 \circ T_1$.